On Correlation Effects and Systemic Risk in Credit Models*

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Abstract

In this paper we establish a family of models where the credit spreads of multiple firms and the term structure of interest rates at any future date can be represented, analytically, in terms of a finite number of state variables. The models make no restrictions on the correlation structure between interest rates and credit spreads. Default correlations among credit spreads of different firms are induced by allowing the intensity rates of different firms to be correlated with each other. In addition, clustering behavior of defaults is obtained by allowing default events to have both temporary and permanent effects on prices of bonds in related industries. Our multifactor models therefore allow us to explore the effects of both types of correlations in interest and credit sensitive contracts. Moreover, since the clustering behavior can be induced by risks common to all bonds in an industry, our models also allows the effects of systemic risk to be closely examined.

This paper investigates Markovian models in the Heath Jarrow Morton (1992) (hereafter HJM) paradigm that can be used to price credit derivatives on both single and multiple names. The models we develop have the following properties. First, they fully incorporate the current riskless term structure information as well as the full credit spread curve information for each firm. Second, the models, being Markovian, permit the riskless and risky credit spread curves to be analytically computed, at any point in time based on a finite collection of state variables. Third, the models allow for arbitrary correlation between riskless interest rates and credit spreads. Fourth, the models include those for which interest rates and credit spreads are non-negative. Fifth, interest rate and credit spread volatilities could be time homogeneous and level dependent. Finally, not only are default rates correlated, but defaults of individual firms could lead to ripple or jump effects in credit spreads of related firms. That is, the models permit defaults to cluster over time due to business cycle effects or due to other specific individual firm default events.

The models we develop belong to the family of reduced form models, where defaults occur as surprise stopping times. In this framework, the default process of risky debt is modeled directly rather than through the asset process for the firm. In addition, assumptions are required regarding the recovery rate in default. Combining the default process and recovery rate with assumptions on the riskless term structure process, leads to models for risky bonds and their derivative products.[1] Many recent studies of corporate debt model the default intensity of each firm as a function of common and firm specific factors. Bakshi, Madan and Zhang (2001), for example, assume intensities are driven by interest rates, and a firm specific factor, such as leverage. Janosi, Jarrow and Yildirim (2001) assume the intensity depends on interest rates and a market index. Duffee (1999) assumes that the intensities depend on factors relating to interest rates alone while Driessen (2003) adds additional common factors for all firms as well as firm specific factors. Many of the models in this area build off the usual affine framework, where the state variables are modeled as jump diffusions.[2] Such models have been very helpful in explaining expected returns and risk premia in the corporate bond market and have led to a better understanding of the elements of risk that are priced in the corporate bond market. In contrast to these types of studies our orientation in this paper is to establish models for pricing interest rate and credit sensitive products, where prices are set relative to given riskless and risky term structures. As such, we are less interested in the sources of risk premia and our entire

[1] One set of models employs a "credit-rating" based approach in which default is depicted through a gradual change in ratings driven by a Markovian transition matrix. Examples of this approach include Das and Tufano (1996), and Jarrow, Lando and Turnbull (1997). Others, such as Duffie and Singleton (1996) and Madan and Unal (1998) model the default process without credit-rating migrations. An alternative approach, referred to as a *structural modeling approach*, follows the lead of Merton (1974), who views the firm's liabilities as contingent claims written on the firm's underlying assets. For excellent discussions of both the reduced form approach and the structural approach see Duffie and Singleton (2003).

[2] See Duffie, Pan and Singleton (1999), for a discussion of these models.

analysis proceeds under the risk neutral measure.

Duffie and Singleton (1999), Schönbucher (2000), and others, have shown how the HJM paradigm can be extended to include risky debt. Specifically, necessary restrictions on the dynamics of drift terms of forward rates and risky forward credit spreads have been identified that permits risky bonds to be priced in an arbitrage free environment. Unfortunately, the resulting dynamics of all riskless forward rates and risky forward credit spreads are not in general Markov in a finite number of state variables. As a result, implementing these models, even via Monte Carlo simulation, is delicate and computationally intensive. The problem is compounded further if the derivative security that needs to be priced depends on the credit spreads of multiple names. In this paper we generate an m-factor model for the riskless term structure and a correlated n-factor model for forward credit spreads, in such a way that riskless and risky bond prices can be recovered analytically in terms of their initial values and a finite collection of underlying state variables.

When the credit spread dynamics are shut down, ($n = 0$), and the number of stochastic drivers for the riskless term structure is set to one, ($m = 1$), then our model reduces to Ritchken and Sankarasubramanian (1995). They identified necessary and sufficient conditions on the volatility structures of forward rates that permit the entire term structure dynamics to be represented by the short rate and a second auxiliary state variable, that fully captures the path dependence inherent in the HJM models. The volatility structures that are admissible here are fairly large and the models nest the Hull and White (1993) term structure model as well as models where volatilities are level dependent. Our new models also nest multi-variate extensions of the Ritchken and Sankarasubramanian model developed by Inui and Kijima (1998).[3] When credit spreads are also considered, i.e., when $n > 0$, the models become much more complex, especially if credit spreads are correlated with interest rates. Our simplest model, consisting of two correlated factors, $m = n = 1$, where the initial riskless and credit spread curves are taken as given, where volatility structures for forward rates and risky forward credit spreads are time homogeneous and level dependent, and where arbitrary correlation between interest rates and credit spreads is allowed, requires 6 state variables. As m and n increase, the number of state variables increases in a way that depends on the structure of volatility and correlation restrictions.

The analysis is trivial when interest rates and credit spreads are uncorrelated. As a result, our analysis would be of little interest if the correlation effects between interest rates and credit spreads had little effect on prices of credit derivatives. Therefore, all our credit derivative pricing examples that we consider are geared towards illustrating how some credit derivative products are extremely sensitive to correlations between interest rate and credit spreads.

[3]For further Markovian models of the riskless term structure see Bhar and Chiarella (1995) and Cheyette (1995).

This paper focuses on models of credit contracts that depend on multiple names. A main issue here is how to model default correlation. This can be accomplished in several ways. Of course correlation can be directly specified through the joint dynamics of the intensities. However, conditional on these intensities, defaults still remain independent. As a result, the effective default correlation that these models offer may not be dramatic. Jarrow and Yu (2001) and Yu (2002) extend these models so that a default can trigger jumps in the intensities of other firms default processes. Duffie and Singleton (1999b) provide an alternative approach that uses separate point processes, some of which trigger joint defaults, while others reflect firm specific defaults. Finally, Schönbucher and Schubert (2002) permit all individual credits to be specified using any simple default intensity models separately for each firm, and then independently building in a dependence structure via a copula function.

In our models, we introduce default dependence by correlating the intensity processes of different firms. In addition, we also permit jumps to occur in the intensities of some firms when particular events occur. In this regard our models are somewhat similar to Jarrow and Yu (2001), and Yu (2002). These *infection* models assume that the intensity process jumps at the time of a default of another firm, and this leads to a repricing of the bond. In contrast, in our approach, we assume there is an impact on prices, somewhat akin to a recovery, and this affects the forward credit spreads, inducing a jump in their values. The price impacts that we permit fall into two types, namely permanent effects, or temporary. With these additions, Markovian models are developed for the term structures of all firms and derivative contracts based on a portfolio of credits can be efficiently computed. We present examples which highlight the importance of adding these impact factors so as to enhance default correlations. In particular, we examine a credit derivative contract that has a payout linked to a portfolio of bonds at the time of first default in the portfolio.

Finally, our models contribute to the literature on systemic risk. In particular, our models permit events to occur which can trigger large interest or credit risk shocks that may permeate through all bonds in a sector. Such models, therefore, may be of interest in studies of value at risk where the impact of small probability events that cause large correlated losses in industries is currently of much interest.

The paper proceeds as follows. In section 1, we discuss a multi-dimensional single risky bond model and review the necessary constraints on drift terms that prevents riskless arbitrage. In section 2 we consider constraints on volatility terms that enables a Markovian representation. In section 3 we extend the analysis to include portfolios of risky bonds where defaults of any one bond could impact, in a temporary or permanent basis, the credit spreads of other bonds. We provide examples to illustrate implementations of the models, as well as to highlight the importance of correlation effects between credit spreads and interest rates, correlation effects between credit spreads, the impact of clustering, and the consequences of systemic risk on pricing. Section 4 concludes the paper.

1 HJM Models for Defaultable Bonds

Let $P(t,T)$ be the price at date t of a pure riskless discount bond that pays \$1 at date T. Then:

$$P(t,T) = e^{-\int_t^T f(t,u)du}, \tag{1}$$

where $f(t,u)$ is the date t forward rate for the future time increment $[u, u+dt]$. We assume that forward rates follow a diffusion of the form:

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dz(t) \text{ given } f(0,T) \quad \forall T \leq T^*, \tag{2}$$

where the drift term $\mu_f(t,T)$ is a predictable process, the volatility term, $\sigma_f(t,T)$, is a predictable $1 \times m$ vector process, $z(t) = (z_1(t), ..., z_m(t))'$ is an m-dimensional standard Wiener process, and T^* is a distant time horizon. Here we assume that $\mu_f(t,T)$ and $\sigma_f(t,T)$ are regular enough to allow differentiation under the integral sign, interchange of the order of integration, partial derivatives with respect to the T variable, and have the property that the resulting bond prices are bounded. The instantaneous spot rate at date t is $r(t) = f(t,t)$.

Now, consider a risky bond. Let $\Pi(t,T)$ represent the date t price of the bond that promises to pay \$1.0 at date T. The time to default is a stopping time, τ, say. Define $N(t) = 1_{\tau \leq t}$. We assume that $N(t)$ has intensity $\lambda(t)$. If a default occurs, we initially assume there is no recovery. Then:

$$\Pi(t,T) = (1 - N(t))e^{-\int_t^T (f(t,u)+\lambda(t,u))du}. \tag{3}$$

Here $\lambda(t,u)$ is the forward rate spread, representing the difference between the defaultable forward rate and the default free forward rate. We assume:

$$d\lambda(t,T) = \mu_\lambda(t,T)dt + \sigma_\lambda(t,T)dw(t) \text{ given } \lambda(0,T) \quad \forall T \leq T^*, \tag{4}$$

where the drift term $\mu_\lambda(t,T)$ is a predictable process, the volatility term, $\sigma_\lambda(t,T)$, is a predictable $1 \times n$ vector, and $w(t) = (w_1(t), ..., w_n(t))'$ is an n-dimensional standard Wiener process.

Similarly, we assume that the drift and volatility terms have the same regularity conditions as the forward rate process. The instantaneous spread is $\lambda(t,t)$. It can be shown that $\lambda(t) = \lambda(t,t)$. Further, assume $E(dz(t)dw(t)') = \Sigma_{m \times n}dt$. Equation (3) can be rewritten as:

$$\Pi(t,T) = S(t,T)P(t,T), \tag{5}$$

where

$$S(t,T) = (1 - N(t))e^{-\int_t^T \lambda(t,u)du}. \tag{6}$$

Applying Ito's rule to equations (1) and (6) we obtain:

$$\begin{aligned}\frac{dP(t,T)}{P(t,T)} &= -dG(t,T) + \frac{1}{2}(dG(t,T))^2 \\ \frac{dS(t,T)}{S(t,T)} &= -dK(t,T) + \frac{1}{2}(dK(t,T))^2 - dN(t)\end{aligned}$$

where $G(t,T) = \int_t^T f(t,u)du$ and $K(t,T) = \int_t^T \lambda(t,u)du$. Using Lemma 1 in the appendix, the dynamics reduce to:

$$\frac{dP(t,T)}{P(t,T)} = \mu_p(t,T)dt + \sigma_p(t,T)dz(t) \tag{7}$$

$$\frac{dS(t,T)}{S(t,T)} = \mu_s(t,T)dt + \sigma_s(t,T)dw(t) - dN(t) \tag{8}$$

where

$$\mu_p(t,T) = r(t) + \frac{1}{2}\sigma_p(t,T)\sigma_p'(t,T) - \int_t^T \mu_f(t,u)du \tag{9}$$

$$\sigma_p(t,T) = -\int_t^T \sigma_f(t,u)du \tag{10}$$

$$\mu_s(t,T) = \lambda(t) + \frac{1}{2}\sigma_s(t,T)\sigma_s'(t,T) - \int_t^T \mu_\lambda(t,u)du \tag{11}$$

$$\sigma_s(t,T) = -\int_t^T \sigma_\lambda(t,u)du. \tag{12}$$

The dynamics of the risky bond are given by:

$$\frac{d\Pi(t,T)}{\Pi(t,T)} = \frac{dP(t,T)}{P(t,T)} + \frac{dS(t,T)}{S(t,T)} + \frac{dP(t,T)}{P(t,T)}\frac{dS(t,T)}{S(t,T)}.$$

Substituting equations (7) and (8) into the above equation leads to:

$$\frac{d\Pi(t,T)}{\Pi(t,T)} = \mu_\Pi(t,T)dt + \sigma_p(t,T)dz(t) + \sigma_s(t,T)dw(t) - dN(t)$$

where

$$\mu_\Pi(t,T) = \mu_p(t,T) + \mu_s(t,T) + \sigma_p(t,T)\Sigma\sigma_s'(t,T) \tag{13}$$

Proposition 1

Assume the dynamics of forward rates and risky forward credit spreads under the risk neutral measure are given by:

$$\begin{aligned}df(t,T) &= \mu_f(t,T)dt + \sigma_f(t,T)dz(t) \text{ given } f(0,T) \quad \forall T \leq T^* \\ d\lambda(t,T) &= \mu_\lambda(t,T)dt + \sigma_\lambda(t,T)dw(t) \text{ given } \lambda(0,T) \quad \forall T \leq T^*.\end{aligned}$$

Then, to avoid arbitrage opportunities, the following drift restrictions must hold.

$$\mu_f(t,T) = |\sigma_p(t,T)|\sigma_f'(t,T) \tag{14}$$

$$\mu_\lambda(t,T) = |\sigma_s(t,T)|\sigma_\lambda'(t,T) - \sigma_p(t,T)\Sigma\sigma_\lambda'(t,T) - \sigma_f(t,T)\Sigma\sigma_s'(t,T) \tag{15}$$

Proof: See Appendix.

Equations (14) and (15) curtail the drift expressions in terms of the volatility structures. The restriction on the drift terms for riskless forward rates under the risk neutral measure were first identified by HJM (1992). The restrictions for risky forward rates were identified by several authors, including Schönbucher (2000). In general, these restrictions imply that the dynamics of riskless and risky bonds are not Markovian in a finite collection of state variables. This creates computational difficulties since the entire riskless and risky term structures have to be stored along all the paths that are generated.

2 Markovian Models

We now consider restrictions on the dynamics of forward rates and risky forward rate spreads that result in the process for risky bonds being Markovian in a few state variables. First define $A(t,T) = (e^{-\int_t^T \kappa_1(u)du}, ..., e^{-\int_t^T \kappa_m(u)du})$ and $\kappa(t) = (\kappa_1(t), ..., \kappa_m(t))$ where $\kappa_i(t), i = 1, ..., m$ are deterministic functions of time t. Then assume

$$\sigma_f(t,T) = \sigma_r(t) \otimes A(t,T) \tag{16}$$

where the operator $\otimes$ represents an element operation of the form:[4]

$$(a_1, ..., a_m) \otimes (b_1, ..., b_m) \equiv (a_1 b_1, ..., a_m b_m),$$

when a and b are vectors, and:

$$(a_1, ..., a_m)' \otimes \Sigma_{m\times n} = \begin{pmatrix} a_1\Sigma_{1.} \\ a_2\Sigma_{2.} \\ \cdots \\ a_m\Sigma_{m.} \end{pmatrix},$$

when $\Sigma_{m\times n}$ is an $m \times n$ matrix with i^{th} row $\Sigma_{i.}$. In equation (16), $\sigma_r(t) = (\sigma_{r1}(t), ..., \sigma_{rm}(t))$ could depend on information on the term structure up to date t. As an example, $\sigma_r(t) = (\sigma[r(t)]^{\gamma_1}, ..., \sigma[r(t)]^{\gamma_m})$.[5] Further, $A(t,T)$ is a deterministic time varying function satisfying the semi group property, $A(t,T) = A(t,s) \otimes A(s,T)$ for $t \le s \le T$.

[4] The operator $\otimes$ used here should not be confused with the Kronecker product.

[5] To avoid exploding rates, one can always curtail the maximum volatility.

With this structure,

$$\sigma_p(t,T) = -\sigma_r(t) \otimes \int_t^T A(t,u)du \equiv -\sigma_r(t) \otimes \alpha(t,T).$$

Notice, that for $0 \leq u \leq t \leq T$, we have:

$$\begin{aligned} \sigma_f(u,T) &= \sigma_r(u) \otimes A(u,t) \otimes A(t,T) && (17) \\ \sigma_p(u,T) &= -\sigma_r(u) \otimes [\alpha(u,t) + \alpha(t,T) \otimes A(u,t)] && (18) \end{aligned}$$

and $A(t,t) = 1$, $\alpha(t,t) = 0$ and $\frac{\partial \alpha(t,T)}{\partial T} = A(t,T)$.

We impose a similar structure for the volatility structure on the forward credit spreads. In particular, define $B(t,T) = (e^{-\int_t^T \theta_1(u)du}, ..., e^{-\int_t^T \theta_n(u)du})$ and $\theta(t) = (\theta_1(t)....\theta_m(t))$ where $\theta_i(t), i = 1, ..., n$ are deterministic functions of time t. Then assume

$$\sigma_\lambda(t,T) = \sigma_\lambda(t) \otimes B(t,T). \quad (19)$$

Here, $\sigma_\lambda(t) = (\sigma_{\lambda 1}(t), ..., \sigma_{\lambda n}(t))$ could depend on information on the term structure up to date t. With this structure,

$$\sigma_s(t,T) = -\sigma_\lambda(t) \otimes \int_t^T B(t,u)du \equiv -\sigma_\lambda(t) \otimes \beta(t,T).$$

Notice, that for $0 \leq u \leq t \leq T$, we have:

$$\begin{aligned} \sigma_\lambda(u,T) &= \sigma_\lambda(u) \otimes B(u,t) \otimes B(t,T) && (20) \\ \sigma_s(u,T) &= -\sigma_\lambda(u) \otimes [\beta(u,t) + \beta(t,T) \otimes B(u,t)] && (21) \end{aligned}$$

and $B(t,t) = 1$, $\beta(t,t) = 0$ and $\frac{\partial \beta(t,T)}{\partial T} = B(t,T)$.

We now provide the main result of this section.

Proposition 2

Given the dynamics of the forward rates and forward credit spreads are:

$$\begin{aligned} df(t,T) &= \mu_f(t,T)dt + \sigma_f(t,T)dz(t) \text{ given } f(0,T) \quad \forall T \leq T^* \\ d\lambda(t,T) &= \mu_\lambda(t,T)dt + \sigma_\lambda(t,T)dw(t) \text{ given } \lambda(0,T) \quad \forall T \leq T^*. \end{aligned}$$

with the volatility structures curtailed as:

$$\begin{aligned} \sigma_f(t,T) &= \sigma_r(t) \otimes A(t,T) \\ \sigma_\lambda(t,T) &= \sigma_\lambda(t) \otimes B(t,T), \end{aligned}$$

then, future forward rates and forward credit spreads can be represented by a Markovian system. In particular:

(i) Forward rates at date t relate to forward rates at date 0 via:

$$f(t,T) = f(0,T) + A(t,T)\psi_1'(t) + (\alpha(t,T) \otimes A(t,T))\psi_2'(t), \quad (22)$$

where the state variables,

$$\begin{aligned}
\psi_1(t) &= \int_0^t (\sigma_r(u) \otimes \sigma_r(u) \otimes \alpha(u,t) \otimes A(u,t))du + \int_0^t \sigma_r(u) \otimes A(u,t) \otimes dz'(u) \\
\psi_2(t) &= \int_0^t (\sigma_r(u) \otimes \sigma_r(u) \otimes A(u,t) \otimes A(u,t))du,
\end{aligned}$$

have dynamics, given by:

$$\begin{aligned}
d\psi_1(t) &= (\psi_2(t) - \kappa(t) \otimes \psi_1(t))dt + \sigma_r(t) \otimes dz'(t) \\
d\psi_2(t) &= (\sigma_r(t) \otimes \sigma_r(t) - 2\kappa(t) \otimes \psi_2(t))dt.
\end{aligned}$$

(ii) Forward credit spreads at date t, relate to their initial values via:

$$\begin{aligned}
\lambda(t,T) &= \lambda(0,T) + \phi_1(t)B'(t,T) + \phi_2(t)(\beta(t,T) \otimes B(t,T))' + \phi_3(t)B'(t,T) + \phi_4(t)A'(t,T) \\
&+ \alpha(t,T)\phi_5(t)B(t,T) + A(t,T)\phi_5(t)\beta(t,T) \quad (23)
\end{aligned}$$

where the state variables,

$$\begin{aligned}
\phi_1(t) &= \int_0^t \sigma_\lambda(u) \otimes \sigma_\lambda(u) \otimes \beta(u,t) \otimes B(u,t)du + \int_0^t \sigma_\lambda(u) \otimes B(u,t) \otimes dw'(u) \\
\phi_2(t) &= \int_0^t \sigma_\lambda(u) \otimes \sigma_\lambda(u) \otimes B(u,t) \otimes B(u,t)du \\
\phi_3(t) &= 1_m \int_0^t (\sigma_r(u) \otimes \alpha(u,t))' \otimes \Sigma \otimes (\sigma_\lambda(u) \otimes B(u,t))du \\
\phi_4(t) &= 1_n \int_0^t (\sigma_\lambda(u) \otimes \beta(u,t))' \otimes \Sigma' \otimes (\sigma_r(u) \otimes A(u,t))du \\
\phi_5(t) &= \int_0^t (\sigma_r(u) \otimes A(u,t))' \otimes \Sigma \otimes \sigma_\lambda(u) \otimes B(u,t)du,
\end{aligned}$$

have dynamics given by:

$$\begin{aligned}
d\phi_1(t) &= (\phi_2(t) - \theta(t) \otimes \phi_1(t))dt + \sigma_\lambda(t) \otimes dw(t) \\
d\phi_2(t) &= (\sigma_\lambda \otimes \sigma_\lambda(t) - 2\theta(t) \otimes \phi_2(t))dt \\
d\phi_3(t) &= [1_m\phi_5(t) - \phi_3(t) \otimes \theta(t)]\,dt \\
d\phi_4(t) &= [1_n\phi_5'(t) - \phi_4(t) \otimes \kappa(t)]\,dt \\
d\phi_5(t) &= [\sigma_r'(t) \otimes \Sigma \otimes \sigma_\lambda(u) - (\kappa'(t) \otimes \phi_5(t) + \phi_5(t) \otimes \theta(t))]\,dt,
\end{aligned}$$

where $1_m = (1, ..., 1)_{1\times m}$.

Proof: See Appendix.

The above proposition allows us to track a finite number of state variables, whose dynamics are Markovian, and then, at any point in time, to be able to reconstruct the entire term structures of riskless and risky rates. Hence European contracts with payouts relating to par rates, credit spreads, yields or any other cash flows based on functions of these curves can readily be priced using Monte Carlo methods. Moreover, for $m = n = 1$ numerical algorithms based on Li, Ritchken and Sankarasubramanian (1995) can be implemented to price American claims.[6]

Since $\psi_1(t)$, $\psi_2(t)$ are of size m, the riskless forward rate dynamics are Markov in $2m$ variables. Since $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ are of size n, $\phi_4(t)$ is of size m and $\phi_5(t)$ is of size $m \times n$, forward credit spreads are Markov in $3n + m + mn$ variables. Note also that $\phi_1(t)$ and $\psi_1(t)$ are the only non-predictable processes. All the other state variables are path statistics that accumulate information based on the realizations of the innovations.[7]

When the stochastic driven process of credit spreads are shut down ($n = 0$) and interest rates are driven by one stochastic driver, ($m = 1$), this model is similar to that of Ritchken and Sankarasubramanian (1995). For the more general case, when $m > 1$ and $n = 0$, the model corresponds to that of Inui and Kijima (1998).

When n is released from 0 we get some new models for risky credit spreads and bond prices. The total number of state variables is $3n + 3m + mn$. Given these state variables, we can reconstruct the riskless and risky term structure at any time. Unfortunately, as m and n increase, the number of state variables increases rather rapidly. For example, if the riskless term structure is driven by $m = 3$ factors and credit spreads are driven by $n = 2$ factors, the total number of state variables is 21. For pricing derivatives on a single name, this number of state variables is not large enough to provide an excessive computational burden. However, when we consider multiple names, this could become an issue.

To simplify the Markovian model, we can either change the structure of the volatility or reduce the dimension of the driven processes. The rest of this section will focus on simplifying the volatility structures further.

2.1 Simplified Volatility Structure

Let $\kappa_i(t) = \kappa(t), i = 1, ..., m$ and $\theta_j(t) = \theta(t), j = 1, ..., n$. With this simplification, the Markovian model simplifies in a fairly dramatic way. Let us redefine the vectors $A(t,T)$ and $B(t,T)$ as

[6] Monte Carlo methods, along the lines of Longstaff and Schwartz (2001) can also be used to price American claims.

[7] If additional restrictions are placed on the volatility assumptions, then it is possible for these path statistic state variables to drop out. For example, if all the volatility structures are in the Vasiceck class, then the path statistic state variables are unnecessary.

the following scalars:

$$
\begin{aligned}
A(t,T) &= e^{-\int_t^T \kappa(u)du}, \quad \alpha(t,T) = \int_t^T A(t,u)du \\
B(t,T) &= e^{-\int_t^T \theta(u)du}, \quad \beta(t,T) = \int_t^T B(t,u)du.
\end{aligned}
$$

Then the volatility structures for forward rates and forward credit spreads can be rewritten as

$$
\sigma_f(t,T) = \sigma_r(t)A(t,T) \tag{24}
$$

$$
\sigma_\lambda(t,T) = \sigma_\lambda(t)B(t,T). \tag{25}
$$

where $\sigma_r(t) = (\sigma_{r1}(t), ..., \sigma_{rm}(t))$ and $\sigma_\lambda(t) = (\sigma_{\lambda 1}(t), ..., \sigma_{\lambda m}(t))$.

Under this definition, we have:

Proposition 3

Given the dynamics of the forward rates and forward credit spreads are:

$$
\begin{aligned}
df(t,T) &= \mu_f(t,T)dt + \sigma_f(t,T)dz(t) \text{ given } f(0,T) \quad \forall T \leq T^* \\
d\lambda(t,T) &= \mu_\lambda(t,T)dt + \sigma_\lambda(t,T)dw(t) \text{ given } \lambda(0,T) \quad \forall T \leq T^*.
\end{aligned}
$$

with the volatility structures curtailed as in equations (24) and (25). Then, future forward rates and forward credit spreads can be represented by a Markovian system. In particular:

(i) The forward rates are Markov in $r(t)$ and $\psi_2(t)$. Forward rates at date t relate to forward rates at date 0 via:

$$
f(t,T) = f(0,T) + A(t,T)[r(t) - f(0,t)] + \alpha(t,T)A(t,T)\psi_2(t) \tag{26}
$$

where

$$
\psi_2(t) = \int_0^t \sigma_r(u)\sigma_r'(u)A^2(t,u)du
$$

and the dynamics of the state variables are:

$$
\begin{aligned}
dr(t) &= [\kappa(t)(f(0,t) - r(t)) + \psi_2(t) + \frac{\partial}{\partial t} f(0,t)]dt + \sigma_r(t)dz(t) \\
d\psi_2(t) &= [\sigma_r(t)\sigma_r'(t) - 2\kappa(t)\psi(t)]dt
\end{aligned}
$$

(ii) Forward credit spreads are Markov in four state variables, $\lambda(t)$, $\phi_2(t)$, $\phi_4(t)$, and $\phi_5(t)$. In particular:

$$
\begin{aligned}
\lambda(t,T) &= \lambda(0,T) + B(t,T)[\lambda(t) - \lambda(0,t)] + \beta(t,T)B(t,T)\phi_2(t) \\
&+ [A(t,T) - B(t,T)]\phi_4(t) + [\alpha(t,T)B(t,T) + \beta(t,T)A(t,T)]\phi_5(t)
\end{aligned} \tag{27}
$$

where

$$\begin{aligned}\phi_2(t) &= \int_0^t \sigma_\lambda(u)\sigma'_\lambda(u)B^2(u,t)du \\ \phi_4(t) &= \int_0^t \sigma_r(u)\Sigma\sigma'_\lambda(u)A(u,t)\beta(u,t)du \\ \phi_5(t) &= \int_0^t \sigma_r(u)\Sigma\sigma'_\lambda(u)A(u,t)B(u,t)du\end{aligned}$$

are now all scalars with dynamics:

$$\begin{aligned}d\phi_2(t) &= [\sigma_\lambda(t)\sigma'_\lambda(t) - 2\theta(t)\phi_2(t)]dt \\ d\phi_4(t) &= [\phi_5(t) - \kappa(t)\phi_4(t)]dt \\ d\phi_5(t) &= [\sigma_r(t)\Sigma\sigma'_\lambda(t) - (\kappa(t)+\theta(t))\phi_5(t)]dt \\ d\lambda(t) &= \mu_\lambda(t)dt + \sigma_\lambda(t)dw(t),\end{aligned}$$

where: $\mu_\lambda(t) = \theta(t)[\lambda(0,t) - \lambda(t)] + \frac{\partial}{\partial t}\lambda(0,t) + \phi_2(t) - (\kappa(t) - \theta(t))\phi_4(t) + 2\phi_5(t)$.

Proof: See Appendix.

When $m = 1, n = 0$, this model is the exact form of the Ritchken and Sankarasubramanian (1995) model. For $m = 1, n = 1$ there are now a total of 6 state variables.[8] As m and n increase beyond 1, the number of state variables remains unchanged. Indeed, under the simplified volatility structures, the multifactor representations for interest rates and credit spreads can both be reduced to one factor representations. An advantage of the models presented here is that forward credit spreads and forward riskless rates can have arbitrary correlation, and interest rates and spreads can be curtailed to be non negative. This stands in contrast to the Vasicek models, where rates can be negative, and the square root Cox Ingersoll Ross type models, where correlations cannot be arbitrary with interest rates remaining nonnegative.

Given an analytical representation for forward rates and risky forward credit spreads, using equations (1) and (5) we can compute bond prices as:

$$\begin{aligned}P(t,T) &= \frac{P(0,T)}{P(0,t)}e^{-(r(t)-f(0,t))\alpha(t,T)-\frac{\alpha^2(t,T)}{2}\psi(t)} \\ \Pi(t,T) &= (1-N(t))\frac{S(0,T)}{S(0,t)}P(t,T)K(t,T)\end{aligned}$$

where

$$\begin{aligned}\ln(K(t,T)) &= -(\lambda(t)-\lambda(0,t))\beta(t,T) - \frac{\beta^2(t,T)}{2}\phi_2(t) \\ &- (\alpha(t,T)-\beta(t,T))\phi_4(t) - \alpha(t,T)\beta(t,T)\phi_5(t).\end{aligned}$$

[8] In the previous section we established Markovian models with $3m + 3n + mn$ state variables. Hence for $m = n = 1$ we have 7 state variables. Actually, for the special case of one factor models of interest rates and spreads, the volatility structures in this section are of equal generality to those in the previous section. For one factor models, the dynamics of $\phi_1(t)$ and $\phi_3(t)$ in the previous section can be combined into the dynamics of $\lambda(t)$, reducing the number of state variables to six.

2.2 The Importance of Interest Rate-Credit Spread Correlation

In this section we illustrate the model with $m = 1$ and $n = 1$ and 6 state variables, by examining the pricing of two specific credit derivatives that have similar designs but very different exposures to correlation risk between interest rates and credit spreads.

Our first contract allows the holder to return the risky bond at date T provided it has not defaulted and in return receive a payment that guarantees a particular spread over Treasury. The defaultable bond matures at date T^* and at date t is priced at $\Pi(t, T^*)$. The expiration date of the option is $T < T^*$ and the predetermined spread over Treasury (strike) is k. In particular, if the firm has not defaulted at date T, the holder can exchange the bond for a price of $e^{-k(T^*-T)}P(T, T^*)$. So this credit spread put (CSP) option has the payoff function:

$$CSP(T, T; T^*) = 1_{\{\tau > T\}} \left(e^{-k(T^*-T)}P(T, T^*) - \Pi(T, T^*)\right)^+ ,$$

where τ is a stopping time of default. Following Lando (1998), the credit spread put option price at time t is:

$$CSP(t, T, T^*) = E_t \left[e^{-\int_t^T r(s)+\lambda(s)ds} \left(e^{-k(T^*-T)}P(T, T^*) - \Pi(T, T^*)\right)^+\right].$$

The value of such a contract clearly depends on the joint dynamics of the riskless and risky term structures.

Our second contract is somewhat similar. In contrast to a payoff that depends on the spread over a default-free bond at expiration, the payoff of this contract guarantees a minimum predetermined yield. In particular, the payoff function of this put option (PO) with strike of yield K and maturity of T on the defaultable bond $\Pi(T, T^*)$ with $T < T^*$ is:

$$PO(T, T, T^*) = 1_{\{\tau > T\}} \left(e^{-K(T^*-T)} - \Pi(T, T^*)\right)^+ .$$

Following Lando (1998), the price of put option on default bond at time t is

$$PO(t, T, T^*) = E_t \left[e^{-\int_t^T r(s)+\lambda(s)ds} \left(e^{-K(T^*-T)} - \Pi(T, T^*)\right)^+\right]$$

The above two contracts, can be very easily priced using Monte Carlo simulation, where we track the dynamics of the state variables and recreate the term structures at the expiration date. In particular, we assume that the riskless term structure is flat, $f(0, t) = 0.05, \forall 0 \le t \le T^*$, $\sigma_r(t) = 0.04\sqrt{r(t)}$, and $\kappa = 0.01$. Further we assume that the credit spread term structure is also flat, $\lambda(0, t) = 0.02, \forall 0 \le t \le T^*$, $\sigma_\lambda = 0.03\sqrt{\lambda(t)}$, and $\theta = 0.01$.

Figure 1 shows the sensitivity of the two options to the correlation between spreads and interest rates. In this figure the underlying bond is a five year discount bond, and the option expires in one year. The strike price of the credit spread option is $k = 0.02$ and the strike of the fixed yield put option is $K = 0.07$.

Figure 1 Here

The figure shows that the two contracts have very different sensitivities to correlation. In particular, the fixed yield put option is much more sensitive to interest rate-credit spread correlation than the fixed credit spread contract.

The bottom panel of Figure 1 shows the prices of the different fixed yield put options as the expiration date increases. The underlying bond is a five year discount bond. The prices are shown for three different correlation levels. The figure clearly illustrates how important it is to accurately access the correlation between interest rates and credit spreads. Ignoring correlation effects can have profound implications for pricing particular derivative contracts.

In a general HJM paradigm, these types of contracts would be much more difficult to price since the Markovian property would not hold, and in the simulations, the entire term structures at each time partition would have to be tracked. In contrast, in the above examples, only 6 state variables need to be tracked.

3 Correlated Defaultable Bonds Case

So far our models for credit derivatives have focused on contracts on single names, and have not taken into account that groups of firms can be highly interdependent, with a single default of a firm possibly creating a cascade of defaults, or downgrades. In this section we consider models involving multiple bonds where default correlation is introduced in two ways. First, we allow the intensities of defaults of different firms to be correlated. Conditioning on these state variables, however, implies that defaults are independent events and default correlation arises only because of the common influence on the intensities. As a result, clustering of defaults is unlikely to obtain. Second, to capture what Jarrow and Yu (2001) refer to as *"industrial organization interdependencies"*, we allow defaults of any one firm to influence the intensities of other firms. Our approach is somewhat similar to that of Yu (2002), who also shows that this type of approach has advantages over copula methods that build in dependence in rather ad-hoc ways whereby spillover or ripple effects cannot easily be handled. Yu models the hazard process directly allowing defaults of one firm to impact the hazard rates of other firms, with jumps. This, then leads to bond prices having jumps. In our approach, we specify the *impact factors* on prices directly and from it we back out the intensity process with jumps. The structural form for our impact factors is fairly flexible. As in Jarrow and Yu (2001), our impact factors could be set up such that there are "primary" or even hypothetical bonds, driven by Cox processes, whose events trigger jumps in the intensities of "secondary" bonds.

In the simplest model the "primary" bond, bond 1 say, could be driven by a Poisson process, N_1, with constant intensity λ_1, and all individual "secondary" bonds could be driven by independent Poisson processes. In particular, bond i, $i = 2, 3, ..., \ell$ is driven by a Poisson process with parameter λ_i. Assume the impact factors are $q_{1j} = 1$ and $q_{jk} = 0$ for $k = 1, 2, 3, ..., \ell$ and $j = 2, 3, ..., \ell$. This implies that if the primary bond defaults, every secondary bond j will lose all its value. However, any secondary bond defaulting would not impact others. For this setup the exact time to default of bond j is τ_j where:

$$\tau_j = \inf\{t > 0 | N_1(t) + N_j(t) > 0\},$$

implying default takes place if either an idiosyncratic or systematic shock strikes the firm. In our more general model, the impact factors need not be so simple. As we will see the shocks could be permanent, or temporary, and the Poisson processes could be replaced by correlated Cox processes.

To make matters specific, assume that there are ℓ "bonds" (actual or pseudo) in the economy that could impact or be impacted by any " default" event. Let $\Pi_i(t, T)$, $i = 1, ..., \ell$, be the date t price of the i^{th} bond that matures at date T. Define a Cox process that is unique for each bond. Let $N_i(0) = 0$ and $\tau_i = \inf\{t > 0 | N_i(t) > 0\}$ $i = 1, ..., \ell$. The risky bond price is given by:

$$\Pi_i(t, T) = \prod_{j=1}^{\ell} (1 - q_{ij}(t - \tau_j) N_j(t)) e^{-\int_t^T (f(t,u) + \lambda_i^c(t,u)) du} \qquad i = 1, ..., \ell \tag{28}$$

Here, $q_{ij}(x)$ is a function of x. As an example, we define $q_{ij}(x) \equiv q_{ij} e^{-\gamma_{ij}(x)}$, where q_{ij} is a constant with $0 \leq q_{ij} \leq 1$and $\gamma_{ij}(x)$ is a nonnegative function of x. Here τ_j is the last default time of bond j. For a zero recovery bond i, $q_{ii} = 1$ and $\gamma_{ii}(x) = 0$. If $\gamma_{ij}(x) = \gamma_{ij} \times (x)$ then $\gamma_{ij} > 0$, implies the impact of a default by bond j on bond i is temporary, with the γ_{ij}, determining the speed of recovery. In contrast if $\gamma_{ij} = 0$, the impact is permanent.

In general, $\lambda_i^c(t, u)$ is no longer the forward credit spread because of the default impact from other "bonds". But if the function $q_{ij}(t - \tau_{ij}) < 1$, $i \neq j$, we can define the function $c_{ij}(\cdot)$ as: $q_{ij}(t - \tau_j) \equiv 1 - e^{-c_{ij}(t - \tau_j)}$, $i \neq j$, or

$$c_{ij}(t - \tau_j) = -\ln(1 - q_{ij}(t - \tau_j)).$$

We can then rewrite equation (28) as

$$\Pi_i(t, T) = (1 - N_i(t)) e^{-\int_t^T (f(t,u) + \lambda_i(t,u)) du} \qquad i = 1, ..., \ell \tag{29}$$

where

$$\lambda_i(t, u) = \lambda_i^c(t, u) + \sum_{j \neq i} c_{ij}(t - \tau_j) N_j(t) \tag{30}$$

now has the interpretation of a forward rate spread.

Further, define

$$d\lambda_i^c(t, T) = \mu_{\lambda_i}(t, T) dt + \sigma_{\lambda_i}(t, T) dw(t) \tag{31}$$

where $\mu_{\lambda_i}(t,T)$ is a predictable process, $\sigma_{\lambda_i}(t,T) = (\sigma_{\lambda_{i1}}(t,T), ..., \sigma_{\lambda_{in}}(t,T))$, is a predictable n dimensional vector process, and $w(t)$ is n-dimensional Brownian motion with $E(dz(t)dw(t)') = \Sigma_{m\times n}dt$. Then the dynamics of the credit spreads for firm i are:

$$d\lambda_i(t,T) = \mu_{\lambda_i}(t,T)dt + \sigma_{\lambda_i}(t,T)dw(t) + \sum_{j\neq i} c_{ij}(0)dN_j(t) + \sum_{j\neq i} N_j(t)dc_{ij}(t-\tau_j) \tag{32}$$

Equation (28) can be rewritten as

$$\Pi_i(t,T) = S_i(t,T)P(t,T) \tag{33}$$

where

$$S_i(t,T) = \prod_{j=1}^{\ell}(1 - q_{ij}(t-\tau_j)N_j(t))e^{-\int_t^T \lambda_i^c(t,u)du}. \tag{34}$$

Applying Ito's rule to equations (1) and (34) we obtain:

$$\frac{dP(t,T)}{P(t,T)} = \mu_p(t,T)dt + \sigma_p(t,T)dz(t) \tag{35}$$

$$\frac{dS_i(t,T)}{S_i(t,T)} = \mu_{s_i}(t,T)dt + \sigma_{s_i}(t,T)dw(t) - \sum_{j=1}^{n} q_{ij}(0)dN_j(t) - \sum_{j=1}^{n} N_j(t)dq_{ij}(t-\tau_j) \tag{36}$$

where

$$\begin{aligned}
\mu_p(t,T) &= r(t) + \frac{1}{2}\sigma_p(t,T)\sigma_p'(t,T) - \int_t^T \mu_f(t,u)du \\
\sigma_p(t,T) &= -\int_t^T \sigma_f(t,u)du \\
\mu_{s_i}(t,T) &= \lambda_i(t) + \frac{1}{2}\sigma_{s_i}(t,T)\sigma_{s_i}'(t,T) - \int_t^T \mu_{\lambda_i}(t,u)du \\
\sigma_{s_i}(t,T) &= -\int_t^T \sigma_{\lambda_i}(t,u)du.
\end{aligned}$$

The dynamics of risky bond i is given by:

$$\frac{d\Pi_i(t,T)}{\Pi_i(t,T)} = \frac{dP(t,T)}{P(t,T)} + \frac{dS_i(t,T)}{S_i(t,T)} + \frac{dP(t,T)}{P(t,T)}\frac{dS_i(t,T)}{S_i(t,T)}.$$

Substituting equations (35) and (36) into the above equation leads to

$$\frac{d\Pi_i(t,T)}{\Pi_i(t,T)} = \mu_{\Pi_i}(t,T)dt + \sigma_p(t,T)dz(t) + \sigma_{s_i}(t,T)dw(t) - \sum_{j=1}^{n} q_{ij}(0)dN_j(t) - \sum_{j=1}^{n} N_j(t)dq_{ij}(t-\tau_j)$$

where

$$\mu_{\Pi_i(t,T)} = \mu_p(t,T) + \mu_{s_i}(t,T) + \sigma_p(t,T)\Sigma\sigma_{s_i}'(t,T) \tag{37}$$

Notice that this restriction is of the same form as the restriction with no impact factors, given by equation (13).

3.1 Markovian Models for Correlated Defaultable Bonds

We now develop a Markovian model by specifying the volatility structure as in Proposition 3. In particular, for defaultable bond i, we assume the volatility structure is:

$$\sigma_{\lambda_i}(t,T) = \sigma_{\lambda_i}(t)e^{-\int_t^T \theta_i(u)du} \equiv \sigma_{\lambda_i}(t)B_i(t,T). \tag{38}$$

Here, the $1 \times n$ vector $\sigma_{\lambda_i}(t)$ could depend on information on the term structure up to date t. With this structure,

$$\sigma_{s_i}(t,T) = -\sigma_{\lambda_i}(t)\int_t^T e^{-\int_t^u \theta_i(x)dx}du \equiv -\sigma_{\lambda_i}(t)\beta_i(t,T).$$

Notice, that for $0 \leq u \leq t \leq T$, we have:

$$\sigma_{\lambda_i}(u,T) = \sigma_{\lambda_i}(u)B_i(u,t)B_i(t,T) \tag{39}$$

$$\sigma_{s_i}(u,T) = -\sigma_{\lambda_i}(u)[\beta_i(u,t) + \beta_i(t,T)B_i(u,t)] \tag{40}$$

and $B_i(t,t) = 1$, $\beta_i(t,t) = 0$ and $\frac{\partial \beta_i(t,T)}{\partial T} = B_i(t,T)$.

Proposition 4

If the risk neutral dynamics of riskless forward rates and $\lambda_i^c(t,T)$ are given by

$$\begin{aligned} df(t,T) &= \mu_f(t,T)dt + \sigma_f(t,T)dz(t) \text{ given } f(0,T) \quad \forall T \leq T^* \\ d\lambda_i^c(t,T) &= \mu_{\lambda_i}(t,T)dt + \sigma_{\lambda_i}(t,T)dw(t) \text{ given } \lambda_i(0,T) \quad \forall T \leq T^*. \end{aligned}$$

with the volatility structures restricted as in equations (24) and (25), then:

(i) Riskless forward rates are linked to riskless forward rates at date 0, by

$$f(t,T) = f(0,T) + A(t,T)[r(t) - f(0,t)] + \alpha(t,T)A(t,T)\psi_2(t)$$

where

$$\psi_2(t) = \int_0^t \sigma_r(u)\sigma_r'(u)A^2(t,u)du$$

and the dynamics of the state variables are defined in Proposition 3.

(ii) Risky forward credit spreads for bond i are given by:

$$\lambda_i(t,u) = \lambda_i^c(t,u) + \sum_{j \neq i} c_{ij}(t-\tau_j)N_j(t)$$

with

$$\begin{aligned} \lambda_i^c(t,T) &= \lambda_i^c(0,T) + B_i(t,T)[\lambda_i^c(t) - \lambda_i^c(0,t)] + \beta_i(t,T)B_i(t,T)\phi_{i2}(t) \\ &+ [A(t,T) - B_i(t,T)]\phi_{i4}(t) + [\alpha(t,T)B_i(t,T) + \beta_i(t,T)A(t,T)]\phi_{i5}(t) \end{aligned} \tag{41}$$

where dynamics of ϕ_{i2}, ϕ_{i4} and ϕ_{i5} are defined in appendix.

Proof: See Appendix.

Under our assumptions, the riskless term structure dynamics are unaffected by the addition of price-impact effects on the risky bonds. Of course, the dynamics of the risky forward credit spreads are affected by the temporary and/or permanent shocks, and their dynamics include jump effects, which are independent of maturity.

Given the above equations, the riskless bond price, $P(t,T)$, and the risky bond price with zero recovery, $\Pi_i(t,T)$, at date t can be linked to their earlier prices at date 0, via the state variables through:

$$\begin{aligned} P(t,T) &= \frac{P(0,T)}{P(0,t)} e^{-(r(t)-f(0,t))\alpha(t,T)-\frac{\alpha^2(t,T)}{2}\psi(t)} \\ \Pi_i(t,T) &= \frac{S_i(0,T)}{S_i(0,t)} P(t,T) K_i(t,T) \end{aligned}$$

where

$$\ln(K_i(t,T)) = -(\lambda_i^c(t)-\lambda_i^c(0,t))\beta_i(t,T)-\frac{\beta_i^2(t,T)}{2}\phi_{i2}(t)-(\alpha(t,T)-\beta_i(t,T))\phi_{i4}(t)-\alpha(t,T)\beta_i(t,T)\phi_{i5}(t).$$

3.2 The Importance of Default Correlation and Systemic Risk

To illustrate the implementation of the model, and the importance of incorporating default correlation and systemic risk, we consider pricing a credit derivative that is sensitive to correlation in intensities as well as to the default impact factors and to a single systemic risk event. In particular, assume the credit event is triggered by the first default of n defaultable bonds, called First-to-Default (FTD) basket. This contract has a payoff at the time of the first default, τ, if $\tau \leq T$; or 0 if $\tau > T$ where T is the maturity date of the contract. In this example, we assume that when default happens the contract holder will get cash flows from two sources. First, for the defaulted bond, the holder will get the payoff which is the difference between a price calculated by a pre-determined yield and the pre-default price; second, for the non-defaulted bonds, the holder receives the sum of the payoff for each bond which is the difference between the price calculated by a pre-determined yield and the price after default. That is, given default on the i^{th} bond, the payoff function is

$$Z_i(\tau) = \frac{1}{n}\left(\left(e^{-K_i(T^*-\tau)} - \Pi_i(\tau-,T^*)\right)^+ + \sum_{j\neq i}^{n}\left(e^{-K_j(T^*-\tau)} - \Pi_j(\tau,T^*)\right)^+\right) 1_{\tau\leq T}$$

where K_ℓ is a pre-determined yield for bond ℓ, $\ell = 1,2,...,n$. Since this contract is triggered by the first default, the default can happen to any one of the n defaultable bonds. So we need to compute the conditional default probability for bond i given default. By Bayes rule, the probability for the ith bond to default at the default time τ, given a default has occurred is:

$$p_i(\tau) = \frac{\lambda_i^c(\tau)}{\lambda^c(t)},$$

where $\lambda^c(t) = \sum_{i=1}^n \lambda_i^c(t)$. Let τ be the first default time and let $Z(\tau)$ be the payout at default. The FTD put option price at time t can be computed as:

$$\begin{aligned}
PUT_{FTD}(t,T,T^*) &= 1_{\tau>t} E_t \left[\exp\left(-\int_t^\tau r(s)ds\right) Z(\tau) \right] \\
&= 1_{\tau>t} E_t \left[E_\tau \left(\exp\left(-\int_t^\tau r(s)ds\right) Z(\tau) \right) \right] \\
&= 1_{\tau>t} E_t \left[\exp\left(-\int_t^\tau r(s)ds\right) E_\tau\left(Z(\tau)\right) \right] \\
&= 1_{\tau>t} E_t \left[\exp\left(-\int_t^\tau r(s)ds\right) \sum_{i=1}^n p_i(\tau) Z_i(\tau) \right] \\
&= 1_{\tau>t} E_t \left[\int_t^T (\sum_{i=1}^n p_i(s) Z_i(s)) \lambda^c(s) \exp\left(-\int_t^s (r(u)+\lambda^c(u))du\right) ds \right] \\
&= 1_{\tau>t} E_t \left[\int_t^T (\sum_{i=1}^n \lambda_i^c(s) Z_i(s)) \exp\left(-\int_t^s (r(u)+\lambda^c(u))du\right) ds \right]
\end{aligned}$$

where the second last equation follows from Lando (1998).

As an illustration assume there are two risky bonds. We will use a two-factor square root model for the riskless term structure and a two-factor model for each credit spread term structure. That is, $m = n = l = 2$. For simplicity, we assume the riskless term structure is flat and forward rates are $f(0,t) = 0.05, \forall 0 \le t \le T^*$. The volatility structure for forward rates is given by:

$$\sigma_r(t) = (0.04\sqrt{r(t)}, 0.03\sqrt{r(t)}), \text{ and } \kappa = 0.1.$$

Further, we assume that both the initial credit spread term structures are flat, $\lambda_1(0,t) = 0.05$ and $\lambda_2(0,t) = 0.03\ \forall 0 \le t \le T^*$. The volatility structures for the forward credit spreads are:

$$\sigma_\lambda(t) = \begin{pmatrix} \sigma_{\lambda_1}(t) \\ \sigma_{\lambda_2}(t) \end{pmatrix} = \begin{pmatrix} 0.05\sqrt{\lambda_1(t)} & 0.03\sqrt{\lambda_1(t)} \\ 0.02\sqrt{\lambda_2(t)} & 0.04\sqrt{\lambda_2(t)} \end{pmatrix},$$

with $\theta = 0.1$. The default impact matrix is asymmetric, and given by:

$$(q_{ij}) = \begin{pmatrix} 1 & 0.02 \\ 0.01 & 1 \end{pmatrix}.$$

As an example, if bond 1 defaults, the impact on bond 2, is to depreciate the price by 2%. Finally, we assume, both bonds have a maturity of 10 years and the "strike" prices are $K_1 = 0.1$ and $K_2 = 0.08$.

In addition, we assume that there is a rare systemic risk event that is modeled as a Poisson process with intensity λ. If this event occurs it triggers a default in both bonds. The default setting of the intensity is 0. That is, unless specified, this systemic event is turned off. When turned on, the impact matrix becomes:

$$(q_{ij}) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0.02 \\ 0 & 0.01 & 1 \end{pmatrix},$$

where the first row now represents the default triggering event. To compute the price of the claim, the interest rate variables, $r(t)$ and $\phi_2(t)$, need to be tracked over time. Given these state variables, and the initial term structure, the entire riskless term structure can be recovered whenever it is needed. Similarly, the eight state variables, $\lambda_i^c(t)$, $\phi_{i2}(t)$, $\phi_{i4}(t)$, and $\phi_{i5}(t)$, $i = 1, 2$, need to be tracked. Given their values, the risky credit spread curves can be computed. In this application the risky money fund, $\exp(\int_t^s r(u) + \lambda^c(u)du)$, also need to be tracked over time.

Figure 2 shows that the prices of FTD contracts for different expiration dates that range from 0 to 10 years. The top panel compares the results for three different symmetric impact factors where $q_{12} = q_{21} = q$. If there are no impact factors, prices are lower. As the symmetric impact factor, q increases, the price of the put option increases.

Figure 2 Here

The second panel shows the impact of changing the default correlation in the intensities between the two bonds. As the correlation increases, default risk becomes less diversifiable, and the price of the default put option increases. The third panel shows the sensitivity of the FTD option when we increase the variance in the credit spread. As this factor increases, the variance expands and option prices increase.

The final panel shows the impact on prices when systemic events are considered. In particular, we permit the Poisson event parameter, λ, to be released from 0, and take on the values of 4% and 8% per year. As this intensity increases, the risk neutral probability of a default increases, and the payout of the option increases.

4 Conclusion

The market for credit derivatives on individual names and on portfolios of names has increased dramatically over the last several years. Pricing credit derivatives relative to given term structures of interest rates and credit spreads is very important. To accomplish this, it is often the case that researchers adopt models where interest rates are uncorrelated with credit spreads.

While this typically does lead to simplifications, it can be the source of large errors. A common valuation approach is to use the HJM paradigm that permits full information on the current term structures to be incorporated into the model. However, without curtailing the structure of volatilities and correlations, this paradigm leads to massive path dependence in pricing and hedging. Our contribution here has been to curtail volatility structures in such a way that the path dependence can be readily captured by a finite set of state variables.

In particular, we extend the HJM Markovian models of riskless bonds to Markovian models of risky bonds. First we establish a multi-factor Markovian model for the case where interest rates are driven by m stochastic drivers and forward credit spreads are driven by n correlated stochastic drivers. Analytical expressions for both risky and riskless term structures were derived in terms of $3n + 3m + mn$ state variables. The resulting models have some desirable properties. First, forward rate volatilities can be level dependent. Second, rates can be curtailed to be non negative. Third, spreads and riskless rates can be correlated in an arbitrary way. Finally, for pricing derivatives, the initial term structures of interest rates and credit spreads are taken as given. The importance of correlation among spreads and riskless rates was highlighted through illustrative examples where rates were always positive, volatilities were level dependent and correlation could easily be adjusted from -1 to $+1$.

This paper also extended the analysis to consider multiple bonds, where default of any one bond could impact the prices of others. To facilitate the fact that defaults could have ripple effects, we introduced default impact factors, which could be temporary or permanent. We then extended the model for pricing risky debt to incorporate these effects. To illustrate the model we considered the price of a contract, which at the time of default of the first bond in a portfolio, had a payout that was linked to the recovery rate of the defaultable bond, as well as on the prices of the remaining bonds that were in the portfolio. The example showed the importance of not only the correlations among the default intensities, but also the effect of the default impact factors in pricing. The example also highlighted the importance of systemic risk, induced by an event triggering multiple defaults in an industry.

The clustering property of defaults is an important property of our model that is consistent with the high degree of correlation of defaults with the business cycle. Our models permit issues of systemic risk to be addressed. In these models, the intensities of the events controlling systemic risk are exogenously provided. As a result, the models can immediately be used to assess the adequacy of value at risk systems during periods of crisis, that trigger a cascade of defaults.

It remains for future research to empirically examine models in this family and perhaps establish how many stochastic drivers are necessary to adequately model the credit spread dynamics.

Appendix

Lemma 1

Let $G(t,T) = \int_t^T h(t,u)du$ where

$$dh(t,u) = \mu_h(t,u)dt + \sigma_h(t,u)dw(t).$$

Then

$$dG(t,T) = \mu_G(t,T)dt + \sigma_G(t,T)dw(t)$$

where

$$\begin{aligned} \mu_G(t,T) &= \int_t^T \mu_h(t,s)ds - h(t) \\ \sigma_G(t,T) &= \int_t^T \sigma_h(t,s)ds. \end{aligned}$$

Proof

The main tool in deriving these results is Fubini's theorem for stochastic integrals. For a proof of the above result see HJM (1992).

Proof of Proposition 1

If the dynamics are specified under the risk neutral measure, to avoid riskless arbitrage:

$$E[\frac{dP(t,T)}{P(t,T)}] = E[\frac{d\Pi(t,T)}{\Pi(t,T)}] = r(t)dt$$

It immediately follows from equation (7) and (9) that, to avoid riskless arbitrage, under this measure

$$\frac{1}{2}\sigma_p(t,T)\sigma_p'(t,T) = \int_t^T \mu_f(t,s)ds$$

Differentiating both sides with respect to T, we obtain the HJM no arbitrage restriction on the drift term, namely:

$$\mu_f(t,T) = |\sigma_p(t,T)|\sigma_f'(t,T) \tag{A.1}$$

Now consider the drift restriction on the riskless bond. We have:

$$\mu_\Pi(t,T) - \lambda(t) = r(t)$$

Substituting equation (13) for $\mu_\Pi(t,T)$, we obtain

$$\mu_p(t,T) + \mu_s(t,T) + \sigma_p(t,T)\Sigma\sigma_s(t,T) - \lambda(t) = r(t).$$

Substituting for $\mu_s(t,T)$, using equation (11), leads to:

$$\int_t^T \mu_\lambda(t,u)du = \frac{1}{2}\sigma_s(t,T)\sigma_s'(t,T) + \sigma_p(t,T)\Sigma\sigma_s'(t,T).$$

Differentiating with respect to T, leads to

$$\mu_\lambda(t,T) = |\sigma_s(t,T)|\sigma'_\lambda(t,T) - \sigma_p(t,T)\Sigma\sigma'_\lambda(t,T) - \sigma_f(t,T)\Sigma\sigma'_s(t,T) \tag{A.2}$$

Proof of Proposition 2

To prove the forward rate expression, substitute equations (17) and (18) into the drift restriction equation, (14). This yields,

$$\mu_f(t,T) = (\sigma_r(u) \otimes A(u,t) \otimes A(t,T) \otimes \sigma_r(u) \otimes [\alpha(u,t) + \alpha(t,T) \otimes A(u,t)]$$

The result follows by substituting the above equation and using equation (18) in the expression:

$$f(t,T) = f(0,T) + \int_0^t \mu_f(u,T)du + \int_0^t \sigma_f(u,t)dw(u)$$

To establish the forward credit spread equation, we begin with:

$$\lambda(t,T) - \lambda(0,T) = \int_0^t \mu_\lambda(u,T)du + \int_0^t \sigma_\lambda(u,T)dw(u). \tag{A.3}$$

Substituting equations (17), (18), (20) and (21) into (11) we obtain:

$$\begin{aligned} \mu_\lambda(u,T) \quad = \quad & \eta_1(u,t)B'(t,T) + \eta_2(u,t)(\beta(t,T) \otimes B(t,T))' + \eta'_3(u,t)B'(t,T) \\ & + A(t,T)\eta_4(u,t) + \alpha(t,T)\eta_5(u,T)B'(t,T) + A(t,T)\eta_5(u,T)\beta'(t,T) \end{aligned} \tag{A.4}$$

where

$$\eta_1(u,t) = \sigma_\lambda(u) \otimes \sigma_\lambda(u) \otimes \beta(u,t) \otimes B(u,t) \tag{A.5}$$

$$\eta_2(u,t) = \sigma_\lambda(u) \otimes \sigma_\lambda(u) \otimes B(u,t) \otimes B(u,t) \tag{A.6}$$

$$\begin{aligned} \eta_3(u,t) &= (\sigma_r(u) \otimes \alpha(u,t))\Sigma \otimes (\sigma_\lambda(u) \otimes B(u,t)) \\ &= 1_m((\sigma_r(u) \otimes \alpha(u,t))' \otimes \Sigma \otimes (\sigma_\lambda(u) \otimes B(u,t)) \end{aligned} \tag{A.7}$$

$$\begin{aligned} \eta_4(u,t) &= (\sigma_\lambda(u) \otimes \beta(u,t))'\Sigma' \otimes (\sigma_r(u) \otimes A(u,t)) \\ &= 1_n((\sigma_\lambda(u) \otimes \beta(u,t))' \otimes \Sigma' \otimes (\sigma_r(u) \otimes A(u,t)) \end{aligned} \tag{A.8}$$

$$\eta_5(u,t) = (\sigma_r(u) \otimes A(u,t))' \otimes \Sigma \otimes (\sigma_\lambda(u) \otimes B(u,t)) \tag{A.9}$$

Here we use the following property in the equations of $\eta_3(u,t)$ and $\eta_4(u,t)$,

$$(x \otimes y)\Sigma = x(y' \otimes \Sigma)$$

where x and y are any m dimensional row vectors and Σ is a $m \times n$ matrix. In particular, we take $x = 1_m$, in equations (A.7) and (A.8).

The result follows by substituting these expressions back into equation (A.3) and defining $\phi_1(t) = \int_0^t \eta_1(u,t)du + \int_0^t \sigma_\lambda(u) \otimes B(u,t) \otimes dw'(u)$ and $\phi_j(t) = \int_0^t \eta_j(u,t)du$, for $j = 2, .., 5$. The dynamics of the state variables follows from Ito's lemma.

Proof of Proposition 3

Under the simplified volatility structure, equation (A.4) can be rewritten as,

$$\begin{aligned}\mu_\lambda(u,T) &= B(t,T)\eta_1(u,t) + \beta(t,T)B(t,T)\eta_2(u,t) + B(t,T)\eta_3(u,t) \\ &\quad + A(t,T)\eta_4(u,t) + [\alpha(t,T)B(t,T) + \beta(t,T)A(t,T)]\eta_5(u,t). \qquad \text{(A.10)}\end{aligned}$$

where

$$\begin{aligned}\eta_1(u,t) &= \sigma_\lambda(u)\sigma'_\lambda(u)\beta(u,t)B(u,t) \\ \eta_2(u,t) &= \sigma_\lambda(u)\sigma'_\lambda(u)B^2(u,t) \\ \eta_3(u,t) &= \sigma_r(u)\Sigma\sigma'_\lambda(u)\alpha(u,t)B(u,t) \\ \eta_4(u,t) &= \sigma_r(u)\Sigma\sigma'_\lambda(u)A(u,t)\beta(u,t) \\ \eta_5(u,t) &= \sigma_r(u)\Sigma\sigma'_\lambda(u)A(u,t)B(u,t)\end{aligned}$$

Further, equation (A.3) can be simplified. Substituting (A.10) into (A.3), we have

$$\begin{aligned}\lambda(t,T) &= \lambda(0,T) + B(t,T)\phi_1(t) + \beta(t,T)B(t,T)\phi_2(t) + B(t,T)\phi_3(t) \\ &+ A(t,T)\phi_4(t) + [\alpha(t,T)B(t,T) + \beta(t,T)A(t,T)]\phi_5(t) \\ &+ B(t,T)\int_0^t \sigma_\lambda(u)B(u,t)dw(u) \qquad \text{(A.11)}\end{aligned}$$

where $\phi_j(t) = \int_0^t \eta_j(u,t)du$, for $j = 1,2,..,5$. Dividing both sides by $B(t,T)$ and grouping all terms involving T on the right hand side, we obtain:

$$\begin{aligned}&\frac{\lambda(t,T)-\lambda(0,T)}{B(t,T)} - \beta(t,T)\phi_2(t) - \frac{A(t,T)}{B(t,T)}\phi_4(t) - [\alpha(t,T) + \beta(t,T)\frac{A(t,T)}{B(t,T)}]\phi_5(t) \\ = \; &\phi_1(t) + \phi_3(t) + \int_0^t \sigma_\lambda(u)B(u,t)dw(u)\end{aligned}$$

Since the right hand side is independent of T, and the left hand side holds for all $T \geq t$, taking $T = t$, we obtain:

$$\begin{aligned}&\frac{\lambda(t,T)-\lambda(0,T)}{B(t,T)} - \beta(t,T)\phi_2(t) - \frac{A(t,T)}{B(t,T)}\phi_4(t) - [\alpha(t,T) + \beta(t,T)\frac{A(t,T)}{B(t,T)})\phi_5(t) \\ = \; &\lambda(t) - \lambda(0,t) - \phi_4(t)\end{aligned}$$

which upon rearranging leads to:

$$\begin{aligned}\lambda(t,T) &= \lambda(0,T) + B(t,T)[\lambda(t) - \lambda(0,t)] + \beta(t,T)B(t,T)\phi_2(t) \\ &+ [A(t,T) - B(t,T)]\phi_4(t) + [\alpha(t,T)B(t,T) + \beta(t,T)A(t,T)]\phi_5(t) \qquad \text{(A.12)}\end{aligned}$$

The forward credit spreads $\lambda(t,T)$ are Markov in the four state variables, $\lambda(t)$, $\phi_2(t)$, $\phi_4(t)$, and $\phi_5(t)$ with dynamics:

$$\begin{aligned}
d\phi_2(t) &= [\sigma_\lambda(t)\sigma'_\lambda(t) - 2\theta(t)\phi_2(t)]dt \\
d\phi_4(t) &= [\phi_5(t) - \kappa(t)\phi_4(t)]dt \\
d\phi_5(t) &= [\sigma_r(t)\Sigma\sigma'_\lambda(t) - (\kappa(t)+\theta(t))\phi_5(t)]dt \\
d\lambda(t) &= \mu_\lambda(t)dt + \sigma_\lambda(t)dw(t)
\end{aligned}$$

The dynamics of the forward rates can be obtained using the same steps as the forward credit spreads and hence is not provided. For the case where $n = 1$ see Ritchken and Sankarasubramanian (1995).

Proof of Proposition 4

Now consider the dynamics of $\lambda_i^c(t,T)$. As in section 2, we have:

$$\begin{aligned}
\lambda_i^c(t,T) &= \lambda_i^c(0,T) + B_i(t,T)[\lambda_i^c(t) - \lambda_i^c(0,t)] + \beta_i(t,T)B_i(t,T)\phi_{i2}(t) \\
&+ [A(t,T) - B_i(t,T)]\phi_{i4}(t) + [\alpha(t,T)B_i(t,T) + \beta_i(t,T)A(t,T)]\phi_{i5}(t) \quad \text{(A.13)}
\end{aligned}$$

Given the relationship of $\lambda_i^c(t,T)$ and $\lambda_i(t,T)$ in (30), the forward credit spreads $\lambda_i(t,T)$ are Markov in default events processes $N_{ij}(t), j = 1, ..., n$, and the four state variables, $\lambda_i^c(t)$, $\phi_{i2}(t)$, $\phi_{i4}(t)$, and $\phi_{i5}(t)$ with dynamics:

$$\begin{aligned}
d\phi_{i2}(t) &= [\sigma_{\lambda_i}(t)\sigma'_{\lambda_i}(t) - 2\theta_i(t)\phi_{i2}(t)]dt \\
d\phi_{i4}(t) &= [\phi_{i5}(t) - \kappa(t)\phi_{i4}(t)]dt \\
d\phi_{i5}(t) &= [\sigma_r(t)\Sigma\sigma'_{\lambda_i}(t) - (\kappa(t)+\theta_i(t))\phi_{i5}(t)]dt \\
d\lambda_i^c(t) &= \mu_{\lambda_i}(t)dt + \sigma_{\lambda_i}(t)dw(t).
\end{aligned}$$

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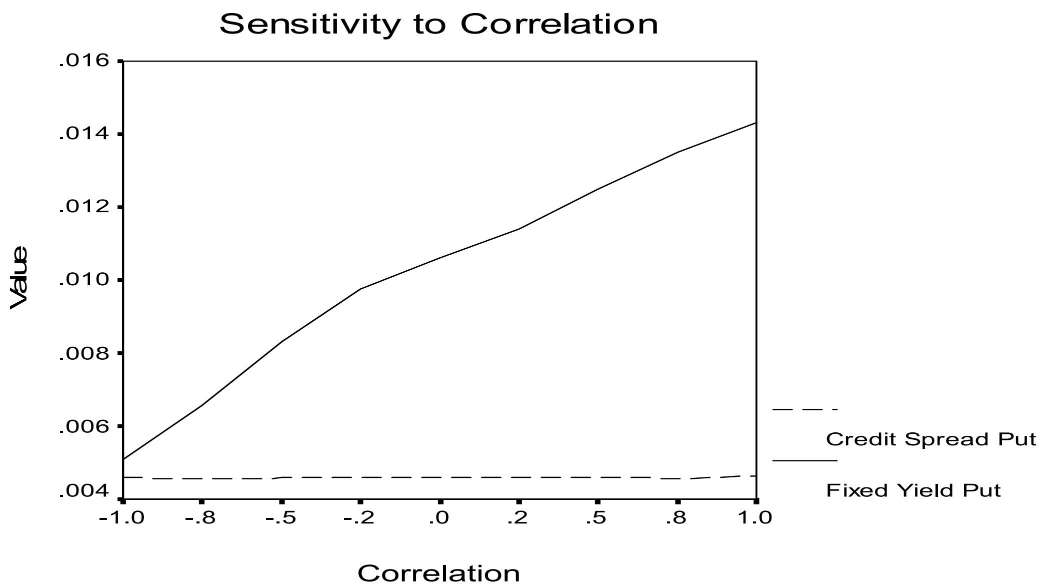
Figure 1
The Effects of Correlation on Prices

The top figure shows the sensitivity of two option contracts to the correlation between credit spreads and interest rates. The underlying bond is a five year corporate discount bond. The option contracts both expire in one year. The credit spread put contract allows the holder to return the bond at a predetermined discount rate of k=2% above the riskless yield rate at maturity. In contrast, the fixed yield put option specifies a fixed strike of 7%, regardless of riskless yields. The initial riskless term structure is flat at 5\% and the initial credit spread curve is flat at 2%. The bottom figure shows the sensitivity of fixed yield puts to differing expiration dates, for three levels of correlation.



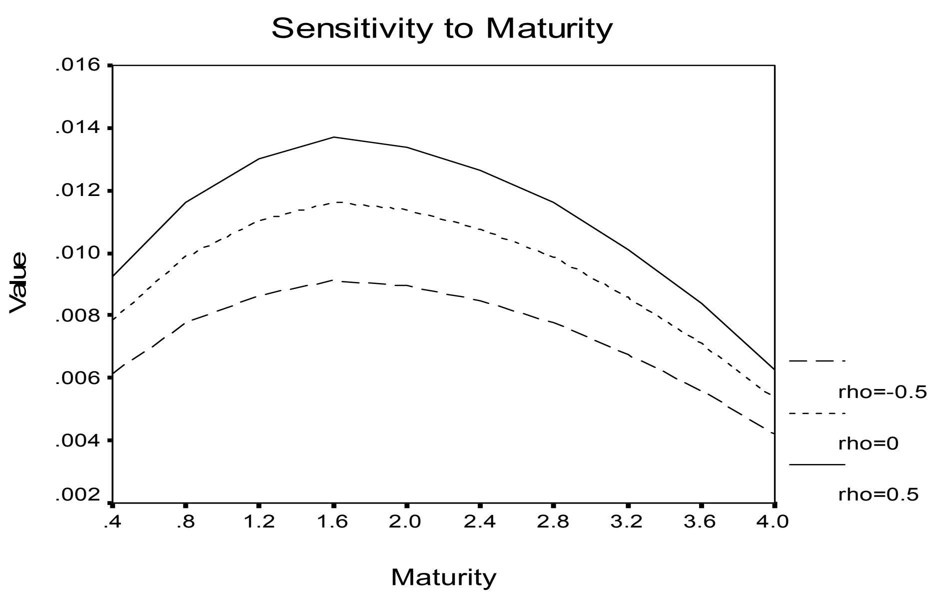


Figure 2
Sensitivity of First to Default Options

The figures show the sensitivity of First To Default Put Options for different expiration dates. The initial credit spread curves are all flat at 5%, for the first, and 3% for the second. The riskless term structure is flat at 5%. The first figure shows the effects of increasing the impact factor, q, which is assumed to be symmetric. The second shows the sensitivity of changing the default correlation among the intensities. The third shows the effects of increasing the variance of the credit spread. The final figure shows the effects of increasing systemic risk.