

# A Pricing Model for Credit Derivatives: Application to Default Swaps and Credit Spreads

Rong Fan \*

Peter Ritchken<sup>†</sup>

August 23, 2001

---

\*Department of Operations Research and Operations Management, Weatherhead School of Management, Case Western Reserve University, 10900 Euclid Ave., Cleveland, Ohio, 44106, Tel: 216-3685009, Fax: 216-368-4776, E-mail: rong.fan@po.cwru.edu

<sup>†</sup>Department of Banking and Finance, Weatherhead School of Management, Case Western Reserve University, 10900 Euclid Ave., Cleveland, OH 44106-7235, Phone: (216) 368-3849, Fax: (216) 368-4776, E-mail: phr@po.cwru.edu

## Abstract

In this article we develop a three factor model for pricing risky bonds. When credit worthiness is not an issue the model collapses to a two factor Heath Jarrow Morton model where forward rate volatilities are humped functions of their maturities and discount bonds are priced at their market values. Risky forward rates are generated by an additional factor. We establish an analytical model for risky bond prices and for a term structure of credit spreads. Our model permits interest rates to be arbitrarily correlated with the short term credit spread. In addition, the credit spread is mean reverting, and, if the parameters of the process are nonnegative, the credit spreads are guaranteed to be positive. The credit spread model is capable of generating an array of credit spread curves similar to those encountered in practice. Models are also developed for a few credit derivatives including default swaps. The fair default swap rate is compared to the credit spread and the difference linked to correlation effects between interest rates and short credit spreads.

This paper presents a simple model of credit spreads that can be used to efficiently implement pricing of credit derivatives on a single name. In our approach, risky bond prices depend on the dynamics of the riskless spot rate and the short credit spread. Since significant information is available regarding the riskless term structure, our model for the riskless term structure is fairly demanding. Specifically, our model for riskless bonds is a two factor model which prices all riskless discount bonds at their market values. Riskless forward rates are imperfectly correlated and their volatility structure is a humped function of their maturities. This feature is consistent with empirical evidence.<sup>1</sup> In contrast to riskless rates, the amount of information available for a particular credit is fairly limited. As a result, our model for credit spreads that we adopt is less demanding. Specifically, we will not require risky discount bonds to be priced to given observables; rather, the risky discount function will be an output of the model, and the parameters of the model will be chosen so that the theoretical prices of a set of risky coupon bonds are priced at values close to their observed market prices. We do require credit spreads to be mean reverting and we can ensure the spreads are non negative with arbitrary correlation with interest rates.

Our model for risky bond prices can be viewed as being comprised of three factors, where one factor drops out if credit risk is not present. The riskless term structure is described by a time varying two factor Heath Jarrow Morton (1992) model, and the credit spread process by a correlated one factor time homogeneous process. Analytical solutions are available for riskless and risky bond prices, conditional of knowledge of the underlying state variables. Since there are just three state variables, implementing these models for pricing credit derivatives is fairly straightforward.

Our model is first developed in discrete time. This is advantageous since in implementing credit derivative models, discrete time partitions are often required, and our model can be directly applied without requiring any adjustments. In addition, in our analysis, we never require a money fund that is continuously earning interest at the instantaneous riskless rate. Rather, our accounts are rolled over at a discrete rate corresponding to the time increment. Since only discrete rates are used, we never require continuous forward rate functions, and hence our analysis is more in tune with the so called market models. Finally, as we shall see, it is possible to develop continuous time results as special limiting cases.

Since our model allows arbitrary correlation between credit and interest rate spreads, it allows us to investigate the importance of correlations in credit derivative contracts. As an example, we explore the impact of correlations on default swap rates. Hull and White (2000) have recently considered the pricing of default swaps. For the case where there is no correlation between interest rates and credit spreads, they show that there is very little difference between

---

<sup>1</sup>For example, see Amin and Morton (1994), Driessen, Klassen and Melenberg (2001), Moraleda and Vorst (1997) and Ritchken and Chuang (1999).

credit spreads and default swap spreads. We extend their results by exploring the precise effects of correlation.

The paper proceeds as follows. In the first section we provide a brief review of the literature on pricing claims sensitive to credit risk and place our models in appropriate context. In the second section we develop our model for the dynamics of riskless rates. This model is a discrete time version of a two factor Heath Jarrow Morton (1992) model, and its diffusion limit has been investigated by Hull and White (1994). In section 3 we add a correlated spread process into the analysis and establish formulae for risky bonds and credit spreads. We also provide illustrations of the term structure of credit spreads that illustrate the flexibility of the model. Finally special cases of the model are considered and the diffusion limits explored. In section 4 we illustrate how the parameters of the model can be estimated using term structure data and risky bond prices of Chase Bank Holding Company. Finally, we consider the pricing of some credit derivative contracts. We price credit default swaps and investigate the potential difference between credit default swap rates and the credit spread. The sensitivity of the gap between these rates to the correlation between spreads and interest rates is explored and our results contrasted with other studies. Section 5 summarizes our results and provides possible extensions.

## 1 Literature Review

Models for pricing of risky debt fall into two categories. The first methodology, referred to as a *structural* modeling approach, follows the lead of Merton (1974), who views the firm's liabilities as contingent claims written on the firm's underlying assets. The typical model starts with a process for the evolution of firm value, and specifies the conditions leading to bankruptcy as well as the payoffs to various parties in the event of bankruptcy. Authors using this approach include Briys, E. and F. de Varenne (1997), Collin-Dufresne and Goldstein (1999), and Longstaff and Schwartz (1995). Structural models are attractive on theoretical grounds, as they link the valuation of financial claims to economic fundamentals. However, they have proved to be hard to implement because of the difficulty in valuing the firm's assets, characterizing and measuring the firm's volatility, and because of the complexity of the capital structure of the firm. Moreover, at the empirical level, these models tend to generate spreads that are too low for high quality borrowers.

Rather than modeling defaults as predictable stopping times that occur when a continuous process reaches a default boundary, a second methodology, referred to as the "*reduced form approach*", views defaults as occurring at surprise stopping times. In this framework, the default process of risky debt is modeled directly rather than through the asset process for the firm. Further, assumptions are made regarding the recovery rate in default. Combining the default process and recovery rate with assumptions on the riskless term structure process, leads to

models for risky debt and their derivative products. One set of models employs a “credit-rating” based approach in which default is depicted through a gradual change in ratings driven by a Markovian transition matrix. Examples of this approach include Das and Tufano (1996) and Jarrow, Lando and Turnbull (1997). Others, such as Duffie and Singleton (1999) and Madan and Unal (1998) and Lando (1998) model the default process without credit-rating migrations.

Duffie-Singleton (1999) consider pricing risky bonds under the assumption that given a default occurs, recovery is proportional to the pre-defaultable market value of the debt. They show that the price of a defaultable bond can be obtained as the martingale expectation of the promised face value and coupons, where all payoffs are discounted by a specific discount rate that embodies the riskless time value, the loss arrival rates and the fractional recovery. Specifically, assume the time to default is generated by a Cox process. The intensity of the process, under the risk neutral measure  $Q$ , is  $\lambda(t)$ . Hence, the chance of default over some small time interval,  $\Delta t$  is  $1 - e^{-\lambda(t)\Delta t} \simeq 1 - (1 - \lambda(t)\Delta t) = \lambda(t)\Delta t$ . Let  $\tau$  be the random variable representing the period in which default takes place. If default occurs, the firm recovers a fraction,  $\phi(t)$ , of the value that the bond would have had, if there had been no default. Let  $P(t, T)$  be the price at period  $t$  of a pure riskless discount bond that pays \$1 at period  $T$ , where each period is of width  $\Delta t$ . Similarly, let  $G(t, T)$  represent the period  $t$  value of a risky bond that promises to pay \$1 at period  $T$ . Then the price of a risky bond with maturity  $T$ ,  $T > t + 1$ , viewed from date  $t$ , assuming the bond has not defaulted, is:

$$\begin{aligned} G(t, T) &= P(t, t + 1)E_t^Q[G(t + 1, T)] \\ &= e^{-r(t)\Delta t}[(1 - \lambda(t)\Delta t) \times E_t^Q[G(t + 1, T)I_{\tau > t+1}] + \\ &\quad \lambda(t)\Delta t \times E_t[G(t + 1, T)I_{\tau = t+1}]] \\ &= e^{-r(t)\Delta t}[(1 - \lambda(t)\Delta t) + \lambda(t)\Delta t\phi(t)]E_t^Q[G(t + 1, T)I_{\tau > t+1}] \end{aligned}$$

Notice that  $[(1 - \lambda(t)\Delta t) + \lambda(t)\Delta t\phi(t)] \simeq e^{-\lambda(t)\Delta t(1 - \phi(t))}$ . Denote  $s(t) = \lambda(t)(1 - \phi(t))$ , then the risky bond can be written as:

$$G(t, T) = e^{-(r(t) + s(t))\Delta t} E_t^Q[G(t + 1, T)I_{\tau > t+1}]$$

Indeed, their valuation formula for risky bonds is identical to the formula for risk-free bonds, with the exception that the riskless rate,  $r(t)$ , is replaced by an adjusted short rate given by  $R(t) = r(t) + s(t)$ , where the spread,  $s(t)$ , reflects the local default rate and the fractional loss rate given a default. The advantage of this approach is that once the higher rate is used as a discount rate, then valuation can proceed as if the claim never defaults. The Duffie-Singleton result makes it possible to transfer all the standard term structure models for default-free bonds, to risky bonds, merely by parameterizing  $R(t)$  instead of  $r(t)$ .

It is possible to establish analytical solutions for risky bond prices under appropriate assumptions. For example, if the dynamics of interest rates and spreads are given by multifactor

Gaussian processes, prices can be obtained. In these models the correlation between the spread and the interest rate can be positive or negative. However, these models have the disadvantage that the short rate and spread can go negative. Multifactor square root processes have also been considered. These models, like the Gaussian models, lead to analytical solutions. Unlike the Gaussian models, here the interest rate and spread remain positive. However, in order to obtain simple analytical solutions, the correlation between interest rates and spreads has to be curtailed to be positive. Recently Bakshi, Madan and Zhang (2001) consider a model where the riskless term structure is generated by a two factor model, as in Jegadeesh and Pennacchi (1996), and a correlated spread process follows an Ornstein-Uhlenbeck process.

In many applications, the riskless term structure and the risky term structure are taken as given and the goal is to price credit derivatives relative to these term structures. In such cases Schonbucher (2000) has shown how the Heath Jarrow Morton paradigm can be extended to include risky debt. Specifically, he identifies the necessary restrictions on the term structure of credit spreads that must exist, if risky bonds are to be priced as if the local expectations hypothesis held. Unfortunately, as in the HJM paradigm, unless riskless and risky forward rate volatilities are curtailed, then implementing these models are delicate since the dynamics of the state variables are not Markov in a finite number of state variables. While it might be reasonable to assume a given term structure for riskless rates, in many applications, it might not make sense to assume the credit spread function is available for a particular firm. In such cases the HJM paradigm might be reasonable for the riskless term structure dynamics, and a simpler process for the credit spreads. For example, it may be reasonable to postulate a mean reverting process for the one period credit spread that is correlated with the short riskless rates which fully characterizes the risky credit spread curve. This is the approach that we adopt.

## 2 The Interest Rate Process

Partition the time interval into increments of width  $\Delta t$  years and label the time periods by consecutive integers. Let  $f(t, T)$  be the forward rate at period  $t$ , for the time period  $[T, T + 1]$ . Hence, expressed in years, the actual time is  $[T\Delta t, (T + 1)\Delta t]$ . The forward rates, under the equivalent martingale measure, are updated as follows:

$$f(t + 1, T) = f(t, T) + \sum_{n=1}^2 [\mu_{f_n}(t, T)\Delta t + \sigma_{f_n}(t, T)\sqrt{\Delta t}Z_{t+1}^{(n)}] \quad (1)$$

where  $\{Z_j^{(n)}, n = 1, 2\}$  are independent standard normal random variables and the volatility structures are given by:

$$\sigma_{f_1}(t, T) = b_1 e^{-\kappa_1(T-t)\Delta t} \quad (2)$$

$$\sigma_{f_2}(t, T) = b_2 e^{-\kappa_1(T-t)\Delta t} + c_2 e^{-\kappa_2(T-t)\Delta t}, \quad (3)$$

These volatility structures imply that the volatility function for forward rates can be a humped function of the maturity, which is consistent with empirical evidence. The drift terms in equation (1) are completely determined by the volatility structures, and are given by the discrete version of the Heath, Jarrow, Merton (1992) restriction, which is:

$$\mu_{fn}(t, T) = \frac{\Delta t}{2} \sigma_{fn}^2(t, T) + \sigma_{fn}(t, T) \sum_{j=t+1}^{T-1} \sigma_{fn}(t, j) \Delta t, \quad n = 1, 2 \quad (4)$$

**Proposition 1**

If the dynamics of the forward rates are given by equation (1), with the volatility structures given by equations (2) and (3), and the initial forward rate curve,  $\{f(0, T) | T = 0, 1, 2, \dots\}$ , given, then the dynamics of the term structure can be represented by a two state variable Markov process.

(i) The dynamics of the state variables,  $r(t)$ , and  $u(t)$  are:

$$\begin{aligned} r(t) &= f(0, t) + \ell(t) + a_1(r(t-1) - f(0, t-1) - \ell(t-1)) \\ &\quad + c_1 u(t) + d_1 \sqrt{\Delta t} \epsilon_t \end{aligned} \quad (5)$$

$$u(t) = a_2 u(t-1) + d_2 \sqrt{\Delta t} \eta_t, \quad (6)$$

where

$$\begin{aligned} \ell(t) &= \frac{b_1^2 a_1^2 (a_1^{2t} - 2a_1^t + 1) \Delta t^2}{2(a_1 - 1)^2} + \frac{1}{2} \left[ \frac{b_2 a_1 (a_1^t - 1) \Delta t}{(a_1 - 1)} + \frac{c_2 a_2 (a_2^t - 1) \Delta t}{(a_2 - 1)} \right]^2 \\ d_1 &= \sqrt{(b_2 a_1 + c_2 a_2)^2 + (b_1 a_1)^2} \\ c_1 &= (a_2 - a_1) c_2 k \\ d_2 &= a_2 / k \end{aligned}$$

and  $a_j = e^{-\kappa_j \Delta t}$  for  $j = 1, 2$ ;  $u(0) = 0$ ;  $\eta_t = Z_t^{(2)}$  and  $\epsilon_t = \rho_1 Z_t^{(2)} + \sqrt{1 - \rho_1^2} Z_t^{(1)}$  are standard normal random variables with correlation,  $\rho_1 = \frac{b_2 a_1 + c_2 a_2}{\sqrt{(b_2 a_1 + c_2 a_2)^2 + (b_1 a_1)^2}}$ . Finally,  $k$  is any fixed constant which for the moment can be set to 1.<sup>2</sup>

(ii) The forward rate,  $f(t, T)$ , at date  $t$  is given by:

$$f(t, T) = f(0, T) + h(t-1, T) + a_1^{T-t} (r(t) - f(0, t) - h(t-1, t)) + c_2 (a_2^{T-t} - a_1^{T-t}) k u(t) \quad (7)$$

where

$$h(t, T) = \sum_{n=1}^2 \sum_{j=0}^t \mu_{fn}(j, T) \Delta t \quad (8)$$

(iii) The price at time  $t$  of a riskfree zero coupon bond that pays \$1 at time  $t+n$ ,  $P(t, t+n)$ , is:

$$P(t, t+n) = e^{-A_n(t) - B_n r(t) \Delta t - C_n u(t) \Delta t} \quad (9)$$

<sup>2</sup>When we investigate a diffusion limit, we will set  $k$  at a level different from 1.

where  $A_1(t) = 0$ ,  $B_1 = 1$ ,  $C_1 = 0$ , and

$$\begin{aligned} B_n &= 1 + B_{n-1}a_1 \\ C_n &= c_1B_{n-1} + a_2C_{n-1} \\ A_n(t) &= A_{n-1}(t+1) + B_{n-1}[f(0, t+1) + \ell(t+1) - a_1(f(0, t) + \ell(t))]\Delta t \\ &\quad - \frac{1}{2}[(B_{n-1}d_1 + C_{n-1}\rho_1d_2)^2\Delta t^3 + (C_{n-1}d_2)^2(1 - \rho_1^2)\Delta t^3] \end{aligned}$$

**Proof:** See Appendix.

In this representation, the dynamics of interest rates are seen to follow a mean reverting process, where the long run average of the short rate is itself stochastic, following its own mean reverting process. Similar processes have been considered by Jegadeesh and Pennacchi (1996), and by Hull and White (1994).

## 2.1 Diffusion Limit of the Two Factor Interest Rate Model

Let  $\Delta r(t) = r(t+1) - r(t)$  and  $\Delta u(t) = u(t+1) - u(t)$  be the changes in the state variables over the  $t^{\text{th}}$  time increment, each of width  $\Delta t$ . Then, using equations (5) and (6), we have, upon rearranging:

$$\begin{aligned} \Delta r(t) &= \left[ \frac{f(0, t+1) - f(0, t)}{\Delta t} \right] \Delta t + \left[ \frac{\ell(t+1) - \ell(t)}{\Delta t} \right] \Delta t \\ &\quad + (a_1 - 1)[r(t) - f(0, t) - \ell(t)] + c_1u(t) + d_1\sqrt{\Delta t}\epsilon_{t+1} \end{aligned} \quad (10)$$

$$\Delta u(t) = (a_2 - 1)u(t) + d_2\sqrt{\Delta t}\eta_{t+1} \quad (11)$$

As  $\Delta t \rightarrow 0$ , this model can be made to converge to the Hull and White two factor term structure model.

### Proposition 2

As  $\Delta t \rightarrow 0$ :

(i) The dynamics of the state variables defined in equation (5) and (6) with  $k = \frac{\kappa_1}{(\kappa_1 - \kappa_2)c_2}$  converge to:

$$dr(t) = (f'(0, t) + \bar{u}'(0, t) + \kappa_1(u(t) - r(t) + f(0, t) + \bar{u}(0, t)))dt + \sigma_1dw_1 \quad (12)$$

$$du(t) = -\kappa_2u(t)dt + \sigma_2dw_2(t) \quad (13)$$

where

$$\begin{aligned} \bar{u}(0, t) &= \frac{b_1^2(1 - e^{-\kappa_1 t})^2}{2\kappa_1^2} + \frac{1}{2} \left[ \frac{b_2(1 - e^{-\kappa_1 t})}{\kappa_1} + \frac{c_2(1 - e^{-\kappa_2 t})}{\kappa_2} \right]^2 \\ Cov[dw_1(t)dw_2(t)] &= \rho dt \end{aligned} \quad (14)$$

$$\begin{aligned}
\sigma_1 &= \sqrt{b_1^2 + (b_2 + c_2)^2} \\
\sigma_2 &= 1/k \\
\rho &= \frac{b_2 + c_2}{\sqrt{b_1^2 + (b_2 + c_2)^2}}
\end{aligned} \tag{15}$$

(ii) The price of a bond at calendar time  $t$  that pays out \$1 at calendar time  $T$  is given by  $P^c(t, T)$  where

$$P^c(t, T) = e^{-\hat{\theta}(t, T) - \beta(t, T)r(t) - \gamma(t, T)u(t)} \tag{16}$$

where

$$\begin{aligned}
\beta(t, T) &= \frac{1}{\kappa_1}(1 - e^{-\kappa_1(T-t)}) \\
\gamma(t, T) &= \frac{\kappa_1}{\kappa_1 - \kappa_2} \left( \frac{1 - e^{-\kappa_2(T-t)}}{\kappa_2} - \frac{1 - e^{-\kappa_1(T-t)}}{\kappa_1} \right) \\
\hat{\theta}(t, T) &= - \int_t^T (\beta(t, s)(f'(0, s) + \bar{u}'(0, s) + \kappa_1 f(0, s) + \kappa_1 \bar{u}(0, s)) ds + \frac{1}{2} \sigma_1^2 \int_t^T \beta^2(t, s) ds \\
&\quad + \frac{1}{2} \sigma_2^2 \int_t^T \gamma^2(t, s) ds + \sigma_1 \sigma_2 \rho \int_t^T \beta(t, s) \gamma(t, s) ds
\end{aligned}$$

**Proof:** See Appendix.

The interest rate follows a two factor mean-reverting process with state variable  $u(t)$  reverting to 0 and the initial forward rate curve that matches the actual observed forward rates. There are a total of 5 parameters, namely,  $\kappa_1, \kappa_2, \sigma_1, \sigma_2$  and  $\rho$ .

### 3 Pricing Risky Bonds

Using the reduced form modeling approach with fractional recovery, implies that to price a risky bond, we need to specify the process of interest rates and for the expected loss rate process, the latter which we shall refer to as the instantaneous credit spread, or credit spread, under the risk neutral measure. In our approach, the interest rate process, under the risk neutral measure, is given by our two factor discrete time model, given in equations (5) and (6), where the initial yield curve is given, and the volatility structure for forward rates is humped. The credit spread process that we adopt differs from the interest rate process in that we do not require the dynamics to incorporate information on an initial term structure of credit risk. The reason for this, is that in many applications a full initial term structure of credit spreads is unavailable or difficult to construct. Rather, we postulate a dynamic for the credit spread, and then calibrate the parameters of the process, so that the model produces theoretical prices of specific risky debt prices that are close to their observed market prices.

Specifically, over a period, the credit spread,  $s(t)$ , is updated as:

$$s(t+1) = \alpha_0 + \alpha_1 s(t) + \alpha_2 (\zeta_{t+1} - \alpha_3)^2$$

where  $\zeta_t$  is a standard normal variable that can be written as:

$$\zeta_t = q_1 Z_t^{(1)} + q_2 Z_t^{(2)} + q_3 Z_t^{(3)}$$

where  $Z_t^{(1)}$ ,  $Z_t^{(2)}$  and  $Z_t^{(3)}$  are independent standard normal variables, with  $q_1^2 + q_2^2 + q_3^2 = 1$ . Notice that  $\eta_t = Z_t^{(2)}$  and  $\epsilon_t = \rho_1 Z_t^{(2)} + \sqrt{1 - \rho_1^2} Z_t^{(1)}$ . Let  $\rho_2 = E[\epsilon_{t+1} \zeta_{t+1}]$  and  $\rho_3 = E[\eta_{t+1} \zeta_{t+1}]$ . We have  $\rho_2 = \rho_1 q_2 + \sqrt{1 - \rho_1^2} q_1$ ,  $\rho_3 = q_2$ . Also, if the constraints  $\alpha_0 \geq 0$ ,  $\alpha_1 \geq 0$ , and  $\alpha_2 \geq 0$  are imposed, then the spread does not become negative. The full model for establishing risky forward rates is then:

$$r(t+1) = f(0, t+1) + \ell(t+1) + a_1(r(t) - f(0, t) - \ell(t)) + c_1 u(t) + d_1 \sqrt{\Delta t} \epsilon_{t+1} \quad (17)$$

$$u(t+1) = a_2 u(t) + d_2 \sqrt{\Delta t} \eta_{t+1} \quad (18)$$

$$s(t+1) = \alpha_0 + \alpha_1 s(t) + \alpha_2 (\zeta_{t+1} - \alpha_3)^2 \quad (19)$$

The dynamics for the spread process are mean reverting, persistent, and for appropriate parameter values, nonnegative. Further, the correlation between the short spread and interest rate,  $\rho_{rs}$ , say, is given by:

$$\rho_{rs} = \frac{-\sqrt{2}\alpha_3\rho_2}{\sqrt{1+2\alpha_3^2}}.$$

Hence,  $\alpha_3$  influences the correlation between interest rates and spreads, as well as the skewness of the spread distribution.

Let  $G(t, T)$  be the value at date  $t$  of a risky zero coupon bond that promises to pay \$1 at time  $T$ . Under the equivalent martingale measure, we have

$$G(t, T)1_{\tau > t} = e^{-(r(t)+s(t))\Delta t} E[G(t+1, T)1_{\tau > t+1}], \quad (20)$$

where  $\tau$  is a random variable representing the time period in which default takes place.

### Proposition 3

*If the interest rate and the instantaneous credit spread, under the risk neutral measure, are given by (17), (18) and (19), then, risky zero coupon bond prices at date  $t$  can be expressed as:*

$$G(t, t+n) = P(t, t+n) e^{-\bar{D}_n s(t) \Delta t - \bar{E}_n}, \quad (21)$$

where

$$\bar{D}_n = 1 + \bar{D}_{n-1} \alpha_1, \quad \bar{D}_1 = 1$$

$$\begin{aligned}
\bar{E}_n &= \bar{E}_{n-1} + \bar{D}_{n-1}(\alpha_0 + \alpha_2\alpha_3^2)\Delta t + \frac{1}{2}\ln(1 + 2\bar{D}_{n-1}\alpha_2\Delta t) \\
&\quad - \frac{(\bar{B}_{n-1}d_1\sqrt{\Delta t}\rho_2 + \bar{C}_{n-1}d_2\sqrt{\Delta t}\rho_3 - 2\bar{D}_{n-1}\alpha_2\alpha_3)^2(\Delta t)^2}{2(1 + 2\bar{D}_{n-1}\alpha_2\Delta t)} + \frac{1}{2}(\bar{B}_{n-1}d_1 + \bar{C}_{n-1}d_2\rho_1)^2(\Delta t)^3 \\
&\quad + \frac{(C_{n-1}d_2)^2(1 - \rho_1^2)\Delta t^3}{2} - \frac{\Delta t^3}{2}(\bar{B}_{n-1}d_1\sqrt{1 - \rho_2^2} + \bar{C}_{n-1}d_2\frac{\rho_1 - \rho_2\rho_3}{\sqrt{1 - \rho_2^2}})^2 \\
&\quad - \frac{\Delta t^3}{2}(\bar{C}_{n-1}d_2\frac{\sqrt{1 - \rho_1^2 - \rho_2^2 - \rho_3^2 + 2\rho_1\rho_2\rho_3}}{\sqrt{1 - \rho_2^2}})^2, \bar{E}_1 = 0
\end{aligned}$$

$$\bar{B}_n = 1 + a_1\bar{B}_{n-1}, \bar{B}_1 = 1$$

$$\bar{C}_n = c_1\bar{B}_{n-1} + a_2\bar{C}_{n-1}, \bar{C}_1 = 0.$$

**Proof :** See Appendix.

Knowing the zero coupon value, we can obtain the term structure of credit spreads, which at date  $t$ , is defined as the difference between the yields of defaultable and default-free bonds, and is give by the expression:

$$\begin{aligned}
s(t, T) &= -\ln(G(t, T))/(T - t) - (-\ln(P(t, T)))/(T - t) \\
&= \frac{1}{T - t}\ln\frac{P(t, T)}{G(t, T)} \\
&= \frac{1}{T - t}(\bar{D}_n s(t)\Delta t + \bar{E}_n)
\end{aligned}$$

It is clear from the above formula that while the credit spread formulation is not explicitly dependent on the level of interest rates, it is influenced by the correlation effects with the interest rate process.

Figure 1 shows three possible shapes of the term structure of credit spreads under different parameterizations. As can be seen, the term structure of credit spreads can be upward sloping, downward sloping or humped. That such flexibility in a model is required is supported by empirical evidence.<sup>3</sup>

Insert Figure 1 Here

---

<sup>3</sup>For example see Duffee (1998), Fons (1994), Jones, Mason and Rosenfield (1984), and Bakshi, Madan and Zhang (2001)

### 3.1 Diffusion Limit of Credit Spreads

We now consider a diffusion limit of the spread process, when  $r(t)$  and  $u(t)$  converge to the stochastic mean reverting Gaussian process discussed before. Let

$$\begin{aligned}\bar{\alpha}_0 \Delta t &= \alpha_0 + \alpha_2(1 + \alpha_3^2) \\ \bar{\alpha}_1 \Delta t &= 1 - \alpha_1 \\ \bar{\alpha}_2 \sqrt{\Delta t} &= \alpha_2 \\ \bar{\alpha}_3 &= \alpha_3.\end{aligned}$$

Then we have:

#### Proposition 4

(i) As  $\Delta t \rightarrow 0$ , the dynamics of the state variables defined in equation (17),(18) and(19) converge to

$$\begin{aligned}dr(t) &= (f'(0, t) + \bar{u}'(0, t) + \kappa_1(u(t) - r(t) + f(0, t) + \bar{u}(0, t))dt + \sigma_1 dw_1(t) \\ du(t) &= -\kappa_2 u(t)dt + \sigma_2 dw_2(t) \\ ds(t) &= (\bar{\alpha}_0 - \bar{\alpha}_1 s(t))dt + \sqrt{2}\bar{\alpha}_2 dw_3(t) - 2\bar{\alpha}_2 \bar{\alpha}_3 dw_4(t)\end{aligned}$$

where,

$$\begin{aligned}\bar{u}(0, t) &= \frac{b_1^2(1 - e^{-\kappa_1 t})^2}{2\kappa_1^2} + \frac{1}{2}\left[\frac{b_2(1 - e^{-\kappa_1 t})}{\kappa_1} + \frac{c_2(1 - e^{-\kappa_2 t})}{\kappa_2}\right]^2 \\ Cov[dw_1(t), dw_2(t)] &= \rho dt \\ Cov[dw_1(t), dw_4(t)] &= \bar{\rho}_2 dt \\ Cov[dw_2(t), dw_4(t)] &= \rho_3 dt \\ Cov[dw_3(t), dw_j(t)] &= 0, \quad j = 1, 2, 4 \\ \bar{\rho}_2 &= \rho q_2 + \sqrt{1 - \rho^2} q_1\end{aligned}$$

(ii) The risky bond price at calender time  $t$  is given by:

$$G^c(t, T) = P^c(t, T)e^{-\bar{\theta}(t, T) - \xi(t, T)s(t)}$$

where

$$\begin{aligned}\xi(t, T) &= \frac{1}{\bar{\alpha}_1}(1 - e^{-\bar{\alpha}_1(T-t)}) \\ \bar{\theta}(t, T) &= -\bar{\alpha}_0 \int_t^T \xi(t, s)ds + \frac{1}{2}\bar{\alpha}_2^2(2 + 4\bar{\alpha}_3^2) \int_t^T \xi^2(t, s)ds - 2\sigma_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\rho}_2 \int_t^T \xi(t, s)\beta(t, s)ds \\ &\quad - 2\sigma_2 \bar{\alpha}_2 \bar{\alpha}_3 q_3 \int_t^T \xi(t, s)\gamma(t, s)ds\end{aligned}$$

**Proof:** See Appendix.

In this model, the diffusion limit of the credit spread dynamics is a mean reverting Gaussian process that is correlated with the instantaneous interest rate and with the process for the long run average of the short rate.

### 3.2 A Special Case of the Credit Spread Model

Consider the case when the credit spread innovation is based on the same innovation as that used in the short rate process. At first glance, this sounds unreasonable, since interest rates could change without any credit effects and vice versa. However, as we shall see, this assumption, leads to a model which is fairly close to the more general model. Now the credit spread follows the dynamics:

$$s(t+1) = \alpha_0 + \alpha_1 s(t) + \alpha_2 (\epsilon_{t+1} - \alpha_3)^2 \quad (22)$$

Notice, that the variable  $\epsilon_t$  is the same as that used in interest rate process. The correlation between interest rates and credit spreads is controlled by the level of  $\alpha_3$ . That is

$$\rho_{rs} = -\frac{2\alpha_3}{\sqrt{2 + 4\alpha_3^2}} \quad (23)$$

When  $\alpha_3$  is positive(negative), the correlation is negative (positive). Notice, that if we were to consider two different firms, this model does not imply that the credit spreads are perfectly correlated. To see this, denote the credit spread of a second firm by:

$$\tilde{s}(t+1) = \tilde{\alpha}_0 + \tilde{\alpha}_1 \tilde{s}(t) + \tilde{\alpha}_2 (\epsilon_{t+1} - \tilde{\alpha}_3)^2 \quad (24)$$

The correlation between  $s(t)$  and  $\tilde{s}(t)$  is:

$$\rho_{s\tilde{s}} = -\frac{2(1 + 2\alpha_3\tilde{\alpha}_3)}{\sqrt{2 + 4\alpha_3^2}\sqrt{2 + 4\tilde{\alpha}_3^2}} \quad (25)$$

The correlation is 1 only when  $\alpha_3$  equals  $\tilde{\alpha}_3$ .

Equation (22) is a special case of equation (19) where  $\rho_2 = 1$  and  $\rho_3 = \rho_1$ . As a result, the prices of risky debt can be obtained immediately from Proposition 3, and the diffusion limits follow from Proposition 4. The results are summarized in the following corollary.

#### Corollary

(i) *If the interest rate and the instantaneous credit spread, under the risk neutral measure, are given by (17), (18) and (22), then, risky zero coupon bond prices at date  $t$  can be expressed as:*

$$G(t, t+n) = P(t, t+n)e^{-\bar{D}_n s(t)\Delta t - \bar{E}_n}, \quad (26)$$

where

$$\begin{aligned}
\bar{D}_n &= 1 + \bar{D}_{n-1}\alpha_1, \quad \bar{D}_1 = 1 \\
\bar{E}_n &= \bar{E}_{n-1} + \bar{D}_{n-1}(\alpha_0 + \alpha_2\alpha_3^2)\Delta t + \frac{1}{2}\ln(1 + 2\bar{D}_{n-1}\alpha_2\Delta t) \\
&\quad - \frac{(\bar{B}_{n-1}d_1\sqrt{\Delta t} + \bar{C}_{n-1}d_2\sqrt{\Delta t}\rho_1 - 2\bar{D}_{n-1}\alpha_2\alpha_3)^2(\Delta t)^2}{2(1 + 2\bar{D}_{n-1}\alpha_2\Delta t)} \\
&\quad + \frac{1}{2}(\bar{B}_{n-1}d_1 + \bar{C}_{n-1}d_2\rho_1)^2(\Delta t)^3, \quad \bar{E}_1 = 0 \\
\bar{B}_n &= 1 + a_1\bar{B}_{n-1}, \quad \bar{B}_1 = 1 \\
\bar{C}_n &= c_1\bar{B}_{n-1} + a_2\bar{C}_{n-1}, \quad \bar{C}_1 = 0.
\end{aligned}$$

(ii) As  $\Delta t \rightarrow 0$ , the dynamics of the state variables defined in equation (17),(18) and(19) converge to

$$dr(t) = (f'(0, t) + \bar{u}'(0, t) + \kappa_1(u(t) - r(t) + f(0, t) + \bar{u}(0, t))dt + \sigma_1dw_1(t) \quad (27)$$

$$du(t) = -\kappa_2u(t)dt + \sigma_2dw_2(t) \quad (28)$$

$$ds(t) = (\bar{\alpha}_0 - \bar{\alpha}_1s(t))dt + \sqrt{2}\bar{\alpha}_2d\bar{w}_3(t) - 2\bar{\alpha}_2\bar{\alpha}_3dw_1(t) \quad (29)$$

where

$$\begin{aligned}
\bar{u}(0, t) &= \frac{b_1^2(1 - e^{-\kappa_1 t})^2}{2\kappa_1^2} + \frac{1}{2}\left[\frac{b_2(1 - e^{-\kappa_1 t})}{\kappa_1} + \frac{c_2(1 - e^{-\kappa_2 t})}{\kappa_2}\right]^2 \\
Cov[dw_1(t), dw_2(t)] &= \rho dt \\
Cov[d\bar{w}_3(t), dw_j(t)] &= 0, \quad j = 1, 2
\end{aligned}$$

And the risky bond price at calendar time  $t$  is given by:

$$G^c(t, T) = P^c(t, T)e^{-\bar{\theta}^s(t, T) - \xi(t, T)s(t)}$$

where

$$\bar{\theta}^s(t, T) = -\bar{\alpha}_0 \int_t^T \xi(t, s)ds + \frac{1}{2}\bar{\alpha}_2^2(2 + 4\bar{\alpha}_3^2) \int_t^T \xi^2(t, s)ds - 2\sigma_1\bar{\alpha}_2\bar{\alpha}_3 \int_t^T \xi(t, s)\beta(t, s)ds$$

Notice that for the limiting process, the credit spreads are driven by a stochastic driver that is separate from the interest rate innovation! Indeed, the diffusion limit here, looks remarkably similar to the diffusion limit of the more general process, where the discrete time credit spreads had their own stochastic driver, separate from the interest rate innovation.

In this model, interest rates follow a two factor mean-reverting process with the auxiliary state variable,  $u(t)$ , reverting to 0. The yield curve automatically matches the observable initial yield curve. The credit spread dynamics add an additional 4 parameters to the process, namely,  $\alpha_0$ ,  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$ . In addition, the initial spread,  $s(0)$ , is not observable and has to be implied out from the data.

## 4 Parameter Estimation

To illustrate the above model, we consider potential term structures of credit spread. First, we estimate the parameters of the two factor interest rate process using term structure data.

For constructing the yield curve, we use futures and swap data. For the short end of the curve (up to 1 year maturity), we use the five nearest futures contracts on any given data. These futures rates are interpolated, and then convexity corrected to obtain the forward rates for 3, 6, 9, and 12 month maturities. The rest of the yield curve out to 5 years is estimated using the forward rates bootstrapped at 6 month intervals from market swap rates. The futures and swap data is obtained from DataStream.

The data for this study consists of USD swaption prices. Specifically, from Datastream, the swaptions data set comprises volatilities of swaptions of maturities 6 months, 1-, 2-, 3-, 4-, and 5-years, with the underlying swap maturities of 1-, 2-, 3-, 4, and 5-years each (in all, there are 30 swaption contracts). As per market convention, a swaption is considered at-the-money when the strike rate equals the swap rate for an equal maturity swap.

Like Amin and Morton (1994), Driessen, Klassen and Melenberg (2001), Longstaff, Santa-Clara and Schwartz (2001), and Moraleda and Pelsser (2000), we estimate model parameters from cross sectional options data. This means that at any date we fit models to the prices of swaptions for different maturities and underlying swap expirations. Our objective function is to minimize the sum of squared percentage errors between theoretical and actual prices using a non-linear least squares procedure.

Using data on September 30<sup>th</sup>, 1999, the following estimates were obtained for our two factor volatility structure:

$$\kappa_1 = 0.044978, \kappa_2 = 3.407608, b_1 = 0.00014383, b_2 = -0.012441, c_2 = 0.018797$$

Figure 2 shows the volatility hump for the forward rate volatilities, and Table 1 shows percentage errors in the 30 estimated swaption prices. Since the bid ask spread of swaptions is at least one half of a Black vol, pricing errors of about 3% or less are deemed reasonable. These above parameter estimates are used as benchmark values.

Figure 2 Here

Table 1 Here

To estimate the remaining parameters of the model requires credit spread information. As an example, we consider eight issues of subordinated debt issued by Chase Bank Holding Company. The terms and prices of these issues on September 30<sup>th</sup> 1999 are shown in Table 2.

Table 2 Here

Adopting a criterion of minimizing the sum of squared percentage errors leads to the following benchmark values for the credit spread parameters:

$$s(0) = 0.0023, \alpha_0 = 0.001814, \alpha_1 = 0.003571, \alpha_2 = 0.0065, \alpha_3 = -0.000427$$

The last two columns of Table 2 shows the theoretical prices and the percentage errors of our prices. These estimates are used as our benchmark case parameters in the analysis that follows.

In the above estimation schemes, we used a time partition of  $\Delta t = 0.125$ . Figure 3 shows how the spread produced by the model converge to its diffusion limit as the time partition  $\Delta t$  goes to 0. This figure, which is fairly typical, shows that there is hardly any difference between the curves when  $\Delta t \leq 0.125$  years.

Insert Figure 3 Here

Figure 4 shows the behavior of the mean reversion parameters on the credit spread. From the limit process of credit spread, we know the speed of mean reversion is  $\bar{\alpha}_1$ . Let  $\kappa = \bar{\alpha}_1$ , we plot the credit spread associated with different  $\kappa$ . Here the instantaneous credit spread,  $s_0$ , that is used is below the long run average. As the mean reversion parameter increases, the credit spread curve steepens.

Insert Figure 4 Here

Figure 5 shows the behavior of the credit spread curve to changes in the volatility, which is  $\bar{\alpha}_2 \sqrt{2 + 4\bar{\alpha}_3^2}$  from the limit process.

Insert Figure 5 Here

## 5 Pricing Credit Swaps and Other Credit Derivatives

There are claims that do not depend on credit spread but do depend on the default event. For example, a default digital put option with expiration date  $T$ , is a contract that has the payoff \$1 at the time of default, provided the default occurs before date  $T$ . Otherwise the contracts pays out nothing. For such contracts, we can no longer model the spread process alone, but rather we need to model the risk neutral hazard rate,  $\lambda(t)$ , say, and the loss rate,  $L(t)$ , say, separately. In what follows we shall assume a constant loss rate, and the default process will

be given by a Cox process where the dynamics of the intensity is given by equation (22). A risky discount bond at date  $t$  with zero recovery value, which promises to pay \$1 at date  $t+n$  is given by  $G^0(t, t+n)$ , where the superscript of zero emphasizes the 0 recovery rate. This value is computed using equation (26), where the spread,  $s(t)$ , is replaced by the hazard value,  $\lambda(t)$ .

It turns out that in order to price many credit derivatives, it is useful to be able to price a particular contract, namely a digital default put, that pays out \$1 only if a credit event occurs in a particular period. Once, this contract is priced several important claims can then readily be priced. We first review the important role of this digital put, and show how it relates to more actively traded products.

### Default Digital Put With Protection Period $[t, t+1]$

Consider a default digital put that at date 0 provides protection over the period  $[t, t+1]$ , and pays out at  $t+1$ . Let  $D_0[t, t+1]$  be its date 0 value. Then at date  $t+1$ , we have

$$D_{t+1}[t, t+1] = \begin{cases} 1 & \text{if } t < \tau < t+1 \\ 0 & \text{if otherwise} \end{cases} \quad (30)$$

The risk neutral probability of defaulting in period  $[t, t+1]$  is:

$$E_0[\pi(t, t+1)]$$

where

$$\pi(t, t+1) = e^{-\sum_{j=0}^{t-1} \lambda(j)\Delta t} (1 - e^{-\lambda(t)\Delta t})$$

is the Poisson probability of defaulting in period  $[t, t+1]$ , conditional on knowledge of the path  $\{\lambda(j)|j = 0, 1, 2, \dots, t\}$  of the default process.

The price of the contract at date 0, is then given by,  $D_0[t, t+1]$ , where:

$$\begin{aligned} D_0[t, t+1] &= E_0 \left[ e^{-\sum_{j=0}^t r(j)\Delta t} 1\pi(t, t+1) \right] \\ &= E_0 \left[ e^{-\sum_{j=0}^{t-1} (r(j)+\lambda(j))\Delta t} (e^{-r(t)} - e^{-(r(t)+\lambda(t))\Delta t}) \right]. \end{aligned} \quad (31)$$

### An American Default Digital Put

An American default digital put with maturity,  $T$ , pays out \$1 at the end of the default period, date  $\tau$ , provided  $\tau < T$ . The price of this contract, is the same as the cost of a strip of the above digitals, that cover all the time periods from date 0 to date T. The cost of this portfolio is  $A_0[0, T]$ , where

$$A_0[0, T] = \sum_{j=0}^{T-1} D_0[j, j+1].$$

This position pays \$1 at the end of the time increment in which it defaults, provided default occurs before the last time period,  $[T - 1, T]$ .

### Default Puts

A default put with maturity  $T$  is an option that pays out the principal less the recovery value of the bond at the end of the default period, provided the default period,  $\tau$ , is prior to the expiration date,  $T$ . Let  $DP_0[0, T]$  represent the value of the default put, at date 0, when the protection period is  $[0, T]$ . Then:

$$DP_{\tau+1}[0, T] = \begin{cases} 1 - rv(\tau) & \text{if } \tau < T \\ 0 & \text{if otherwise} \end{cases} \quad (32)$$

where  $rv(\tau)$  is the recovery value of the bond. The following simple arbitrage-free argument, see for example, Schonbucher (2000), shows that to avoid riskless arbitrage, the price of a default put should equal the price of an American digital put minus the price differential between the risky bond and the risky bond with zero recovery.

Consider a portfolio consisting of 1 defaultable bond,  $G(0, T)$  and 1 default put. The payout of this portfolio is shown below:

Strategy	Current Value	$\tau < T$	$\tau > T$
Buy 1 Risky Bond	$G(0, T)$	$rv(\tau)e^{r(T-\tau)\Delta t}$	1
Buy 1 Default Put	$DP_0[0, T]$	$(1 - rv(\tau))e^{r(T-\tau)\Delta t}$	0
$G(0, T) + DP_0[0, T]$		$1e^{r(T-\tau)\Delta t}$	1

The payoff of this portfolio can be replicated by the payout of the following portfolio:

- 1 risky bond with zero recovery,  $G^0(0, T)$ .
- 1 risky American digital put,  $D_0[0, T]$ .

The payouts of this portfolio are:

Strategy	Current Value	$\tau < T$	$\tau > T$
Buy 1 Risky Bond with zero recovery	$G^0(0, T)$	0	1
Buy 1 Default digital Put	$D_0[0, T]$	$(1)e^{r(T-\tau)\Delta t}$	0
$G^0(0, T) + A_0[0, T]$		$1e^{r(T-\tau)\Delta t}$	1

Since these two portfolios give the same value at date  $T$ , to avoid arbitrage their date 0 values must be the same. Hence:

$$G(0, T) + DP_0[0, T] = G^0(0, T) + A_0[0, T]$$

and, hence

$$DP_0[0, T] = A_0[0, T] - [G(0, T) - G^0(0, T)]$$

Hence, if we could value a default digital put with protection period  $[t, t+1]$ , then we could price a default put option on a zero coupon risky discount bond.

Actually, if the default put is on a defaultable *coupon* bond, the above result still holds true as long as we replace the zero coupon bonds with the appropriate coupon bonds. That is:

$$DP_0[0, T] = A_0[0, T] - [\bar{G}(0, T) - \bar{G}^0(0, T)]$$

where  $DP_0[0, T]$  is now the date 0 price of the default put option on a coupon bond priced at  $\bar{G}(0, T)$ , and  $\bar{G}^0(0, T)$  is the price of the same coupon bond, priced under the assumption of zero recovery.

## Default Swaps

A default swap is a form of default insurance contract. The buyer makes a periodic payment, quoted as a percentage of the notional amount, per year. The payments continue until either the expiration date, or until a default event by the underlying reference institution. If there is a default, the buyer delivers the underlying bond to the seller in exchange for par value.

When there is a default most default swaps require the buyer to pay an accrued premium based on the time between default and the last periodic payment. Also, the seller is only required to pay the par value for the bond less the recovery value. The coupon, or accrued interest is not included. In what follows we assume the notional is \$1 and that the payment dates are every  $K$  periods at dates  $t_i = K \times i\Delta t$ , for  $i = 1, 2, \dots, n$ , say, where  $t_n = T$ . The default swap rate is  $s$  per year or  $\$s^* = s \times K\Delta t$  per payment date. The total number of periods for the default swap is  $N^* = K \times n$ .

The differences between the default put and the default swap are two fold. First, for default puts, the full premium is paid up front, while in the case of the swap, the payment is made in consecutive payment dates up to default. Second, for default swaps, upon default, the payment is not  $1 - rv$ , but  $1 - rv - A^s(\tau)$ , where  $A^s(\tau)$  is the accrued premium, based on the time since the previous premium payment. That is:

$$A^s(\tau) = s(\tau - t^*)\Delta t$$

where  $t^*$  is the last date before default, where the buyer made a cash payment.

If swap payments of size  $s_1$  were made every time increment,  $\Delta t$ , then the payment received at the end of a default period is  $(1 - rv - s_1\Delta t)$ . For this unusual default swap we can find the fair swap rate fairly easily. Specifically, from our earlier analysis, a default put option with this payout would be priced at  $DP_0^*[0, T]$ , say, where  $DP_0^*[0, T] = (1 - s_1\Delta t)A_0[0, T] - [\bar{G}(0, T) -$

$\bar{G}^0(0, T]$ . Hence, to prevent arbitrage, we must have

$$DP_0^*[0, T] = \sum_{j=1}^{N^*} G^0(0, j) s_1 \Delta t.$$

Substituting for  $DP_0^*[0, T]$ , we obtain the fair swap rate as:

$$s_1 = \frac{A_0[0, T] - [\bar{G}(0, T) - \bar{G}^0(0, T)]}{(\sum_{j=1}^{N^*} G^0(0, j) + A_0[0, T]) \Delta t}.$$

Since the actual payments dates of most default swaps are not every time increment, but rather every  $K$  time increments, we have:

$$\sum_{j=1}^n G^0(0, t_j) s_K K \Delta t = \sum_{j=1}^{N^*} G^0(0, j) s_1 \Delta t,$$

where  $s_K$  is the fair swap rate, with periodic payments every  $K$  time increments. From the above equation, we have:

$$s_K = \frac{\sum_{j=1}^{N^*} G^0(0, j) s_1}{\sum_{j=1}^n G^0(0, t_j) K}$$

The American digital default contract, the American default put, and the fair credit spread, can then all be priced, once the price of the digital default put for the period  $[t, t + 1]$  can be priced. Proposition 5 gives us the value of the default free digital put.

### Proposition 5

*If the interest rate and the intensity of the Cox process,  $\lambda_t$ , under the risk neutral measure, are given by (17), (18) and (22), then, the default digital put at date  $t$  can be expressed as:*

$$D_0(t, t + 1) = e^{-X_t r(0) \Delta t - Y_t \lambda(0) \Delta t - W_t u(0) \Delta t - M(t, t)} - G^0(0, t + 1), \quad (33)$$

where

$$\begin{aligned} X_t &= 1 + a_1 X_{t-1}, \quad X_0 = 1 \\ Y_t &= 1 + \alpha_1 Y_{t-1}, \quad Y_0 = 0 \\ W_t &= c_1 X_{t-1} + a_2 W_{t-1}, \quad W_0 = 0. \\ M(t, t) &= M(t-1, t) + X_{t-1} [f(0, 1) + \ell(1) - a_1 (f(0, 0) - \ell(0))] \Delta t + Y_{t-1} (\alpha_0 + \alpha_2 \alpha_3^2) \Delta t \\ &\quad + \frac{1}{2} \ln(1 + 2Y_{t-1} \alpha_2 \Delta t) - \frac{(X_{t-1} d_1 \sqrt{\Delta t} + W_{t-1} d_2 \rho_1 - 2Y_{t-1} \alpha_2 \alpha_3)^2 (\Delta t)^2}{2(1 + 2Y_{t-1} \alpha_2 \Delta t)} \\ &\quad - \frac{1}{2} W_{t-1}^2 (1 - \rho_1^2) d_2^2 (\Delta t)^3, \quad M(0, t) = 0 \end{aligned}$$

**Proof :** See Appendix.

## 6 Differences Between Default Swap Rates and Credit Spreads

Proposition 5 allows us to establish the fair default swap rate, and to compare it with appropriate rates obtained from the term structure of credit spreads. Hull and White (2000) among others, discuss why the credit spread for a particular corporate bond, and the default swap rate for a contract with the same maturity, might be different.

To see this, first consider a world where interest rates are certain and the yield curve flat. In this case, a riskless coupon bond trading at par will continue to trade at par at coupon dates. Let  $x$  be the coupon rate for such a bond. Let  $y$  be the par rate for a risky bond of the same maturity. Since credit risk is stochastic the future value of this bond can vary substantially from par. Now consider a portfolio containing this risky bond and a default swap with swap rate  $s$ . If there is no default, the net cash flows are  $y - s$  per year until maturity. Now consider what happens if a default occurs at date  $\tau$ , say, where  $t < \tau < t + 1$ , for some  $t$ . Recall, that upon default the protection buyer only receives the principal, not the accrued interest, in return for the defaulted bond. Further, the protection buyer has to pay the accrued swap payment. The net cash flow is  $rv + 1 - rv - A^s(\tau) = 1 - A^s(\tau)$ , where  $A^s(\tau)$  is the accrued default swap premium owed to the insurer. The value of the riskless bond at date  $\tau$ , will be  $1 + A^f(\tau)$ , where  $A^f(\tau)$  is the accrued interest, given by  $A^f(\tau) = x(\tau - t)$ . If  $y - s < x$ , then arbitrage opportunities would be available as the riskless bond dominates the risky bond/default swap. Hence, for this stylized situation, the default swap rate should be less than the credit spread,  $s < y - x$ .

If the initial yield curve was certain, but decreasing, rather than flat, then the original par coupon riskless bond would increase over time, and, at the time of default, be worth more than par. In this case, if a default occurs, the portfolio of the defaulted bond and the credit swap would be worse off, relative to the strategy of buying a riskless par bond. In this case, to avoid arbitrage,  $s < y - x$ . However, if the initial yield curve is upward sloping, and interest rates certain, then upon default, the receipt of  $1 - A^s(t)$  dollars, might be more than sufficient to purchase the riskless bond. In this case, we can not be certain that  $s < y - x$ . In summary, in a world of no interest rate uncertainty, the gap between the par credit spread and default swap rate is therefore determined by the shape of the term structure and the nature of the credit risk.<sup>4</sup>

As an example, assume there is no interest rate uncertainty, and that the yield curve is flat at  $r$ . Further, assume defaults occur according to a Poisson process, rather than a Cox process and that the recovery rate is  $\phi$ , say. Then the short credit spread is  $s = \lambda \times (1 - \phi)$ , where  $\lambda$ ,

---

<sup>4</sup>The relationship of default swap rates with credit spreads can be made a bit more precise by considering the default swap rate relative to the credit spread of a floating rate bond. In particular, if one compares a portfolio of a risky floating rate bond, originally issued at par, together with a credit default swap, versus a riskless par bond of the same maturity, then one can conclude that the default swap rate should be set below the credit spread of a floating risky bond.

is the constant hazard rate. For this case, the credit spread curve is flat and the default swap rate curve is also flat and slightly lower.<sup>5</sup>

When interest and hazard rates are stochastic and uncorrelated the relationship between credit spreads and default swap rates is more complex. Figure 6 shows a case where for short term contracts the credit spread exceeds the default swap rate, while for longer dated contracts the default swap rates is higher. This example illustrates that the difference between the two spreads could be large even when the correlation is zero. For example, for a ten year contract the default swap rate is about 33% higher than the credit spread of a risky bond issued at par.

Figure 6 Here

It can be shown that the difference between the two rates is insensitive to the assumption on the recovery rate. That is, increasing the recovery rate decreases the credit spread by the same amount as its effect on the default swap rate.

When, interest rates are uncertain and correlated with the credit risk, then the magnitude of the differences between the credit spreads and default swap rates become more complex. For example, take the case where the original yield curve is originally flat. If the short credit spread is highly correlated with interest rates, then unanticipated rising interest rates lower the price of the original riskless par coupon bond. At the same time, the credit spread most likely has increased, reflecting an increased likelihood of default. In this case, the portfolio of the risky bond and the credit swap, will most likely benefit, relative to the alternative strategy of purchasing a riskless bond. Specifically, in the event of a default,  $1 - A^s(t)$  dollars are obtained that may be used to purchase riskless bonds. Further, if interest rates have increased significantly, then multiple bonds can be purchased. On the other hand, if rates fall, the likelihood of a default falls, and the cash flows of  $y - s$  are more likely to continue. While it is true that if a default occurs, the risky bond/credit spread portfolio is at a disadvantage relative to the riskless asset, due to correlation, the likelihood of this scenario is reduced. Hence, overall, as correlation increases, the benefits of the credit swap increase, and so to be compensated for this, the default swap rate might be expected to increase relative to the credit spread.<sup>6</sup>

Figure 7 compares the default rates with the credit spreads for a 5 year contract, where the underlying risky coupon bond is priced at par.

Figure 7 Here

---

<sup>5</sup>For this case, the difference between the two rates is typically within one basis point.

<sup>6</sup>This argument is somewhat similar to the argument used for analysing the difference between forwards and futures when interest rates are uncertain and correlated with the underlying.

In this example, increasing the correlation, keeping other factors the same, leads to the default swap rate increasing faster than the credit spread.<sup>7</sup>

In Figure 8 the difference between the default swap rates and credit spreads are shown for a few different maturity contracts. As the figure illustrates, the slope of the curves, depicting the gap between the two spreads, are slightly increasing as the maturity of the contracts increase. For example, for our case parameters, the difference between default and credit spreads changes about 3 basis points for a 5 year contract, as the correlation changes from -1 to +1. Figure 9 shows the percentage differences. This figure shows that the default swap rates can be significantly understated if they are taken to be equal to credit spreads.<sup>8</sup>

Figures 8 and 9 Here

Of course, if the initial term structure is not flat, then, multiple effects are present and the overall relationship between credit spreads and default swap rates becomes less predictable. Fortunately, with our model for credit spreads and credit swaps available, the magnitude of these differences can be assessed.

## 7 Conclusion

In this article we have developed a three factor model for pricing risky bonds. When creditworthiness is not an issue the model collapses to a two factor Heath Jarrow Morton model where volatilities are humped functions of their maturities and discount bonds are priced at their market values. When credit risk is an issue, an analytical representation for the credit spread is developed. The short credit spread is mean reverting, and, if the parameters of the process are suitably curtailed to be nonnegative, then the spreads are guaranteed to be positive. In our model, interest rates can be arbitrarily correlated with the short spread. A special case of the general model was developed and its properties explored. In this special case, the innovation for the interest rate, was also used as the innovation for the short spread. Interestingly, the diffusion limit of the credit spread process contained an additional stochastic driver, and the model had a form similar to the more general model where the short spread process had its own dynamics.

The credit spread model appears to be capable of generating credit spread curves similar to what are encountered in practice. We illustrated how parameters for the model could be

---

<sup>7</sup>Notice that just increasing the correlation via  $\alpha_3$  changes the mean and variance of the credit spread process. Hence, the parameters  $\alpha_1$  and  $\alpha_2$  are adjusted so as to keep the long run mean and volatility constant. This ensures that the effect we observe is due to correlation effects alone.

<sup>8</sup>The results we obtain do vary slightly when different parameter values are used, but the direction of the results remain unchanged.

estimated. The model, designed to match the riskless term structure, produced swaption prices that were typically within a bid ask spread and the subordinated debt prices were fairly precise. Given a calibrated model, we then illustrated how some credit derivatives could be priced. In particular, we focused on simple default swaps. We developed a pricing model for default swaps, and used the model to assess the difference between credit spreads and default swap rates. The analysis for the difference between these two rates follow along similar lines to the differences between forwards and futures, and the important role of correlation was documented. We showed that as the correlation between interest rates and spreads increased, the gap between the default swap and credit spread increased. Also, the gap widened with the maturity of the contract. Since the gap could be over 2% of the credit spread, in general, using the credit spread as a proxy for the default swap rate may not be appropriate.

With our model in hand, we can easily establish algorithms for pricing American credit derivatives. Since the dynamics of the state variables are Markov, path dependence issues, that typically plague the implementation of Heath Jarrow Morton models are not present. Further, simulation models, along the lines of Longstaff and Schwartz (2000) make pricing these contracts permissible.

In many markets the default swaps may provide a more active market than the underlying risky bonds. In this case the parameter estimates can be backed out from the default swap rates, and then used to assess the credit spreads. In our empirical example, we first estimated the risk free parameters, using swaption contracts in conjunction with term structure data, and then subsequently estimated the credit spreads. It remains for future research to conduct the estimation process simultaneously. It also remains for future research to test these models and to assess whether the credit spread process can be modeled by our simple level independent structure, or whether we need to add in more level dependence.

## References

- Amin, K., and A. Morton, 1994, "Implied volatility functions in arbitrage-free term structure models," *Journal of Financial Economics*, 35, 141-180.
- Bakshi G., D. Madan, and F. Zhang, 2001, "Investigating the Source of Default Risk: Lesson from Empirically Evaluating Credit Risk Models", Working paper, Department of Finance, University of Maryland.
- Briys, E. and F. de Varenne, 1997, "Valuing risky fixed rate debt: An extension, *Journal of Financial and Quantitative Analysis*, 32 (2), 239-248.
- Collin-Dufresne, P. and R. Goldstein, 1999, "Do credit spreads reflect stationary leverage? Reconciling structural and reduced-form frameworks, *Journal of Finance*, (forthcoming).
- Das, S. and P. Tufano, 1996, "Pricing credit-sensitive debt when interest rates, credit ratings and credit spreads are stochastic, *Jouranal of Financial Engineering*, 5, 161-198.
- Driessen, J., P. Klaasen, and B. Melenberg, 2001, "The Performance of Multi-factor Term Structure Models for Pricing and Hedging Caps and Swaptions", Working Paper, Department of Econometrics, Tilburg University.
- Duffee, G., 1998, "The relation between treasury yields and corporate bond yield spreads, *Journal of Finance*, 53, 2225-2242.
- Duffie, D., 1999, "Credit Swap Valuation," *Financial Analysts Journal*, 73-87.
- Duffie, D. and K. Singleton, 1999, "Modeling term structures of default risky bonds, *Review of Financial Studies*, 12 (4), 687-720.
- Fons, J., 1994, "Using default rates to model the term structure of credit risk, *Financial Analysts Journal*, September-October, 25-32.
- Heath, D., R. Jarrow, and A. Morton, 1992, "Bond pricing and The Term Structure of Interest Rates," *Econometrica*, 60, 77-105.
- Hull, J., and A. White, 1994, "Numerical Procedures for Implementing Term Structure Models II: Two Factor Models," *Journal of Derivatives*, 2, 37-48.
- Hull, J., and A. White, 2000, "Valuing Credit Default Swaps 1:No Counterparty Credit Risk", *The Journal of Derivatives*, 29-40
- Jegadeesh N., and G. Pennacchi (1996), "The Behavior of Interest Rates Implied by the Term Structure of Eurodollar Futures", *Journal of Money, Credit, and Banking*, 28, 426-446.
- Jarrow, R., D. Lando, and S. Turnbull, 1997, "A Markov model for the term structure of credit risk spreads, *Review of Financial Studies*, 10, 481-523.

- Jarrow, R. and S. Turnbull, 1995, "Pricing derivatives on financial securities subject to credit risk, *Journal of Finance*, 50, 53-85.
- Jones, E., S. Mason, and E. Rosenfeld, 1984, "Contingent claims analysis of corporate capital structures: An empirical investigation, *Journal of Finance*, 611-625.
- Lando, D., 1998, "On Cox processes and credit risky securities, *Review of Derivatives Research*, 2, 99-120.
- Longstaff, F. and E. Schwartz, 1995, "A simple approach to valuing risky fixed and floating rate debt, *Journal of Finance*, 50, 789-819.
- Longstaff, F. and E. Schwartz, 1995, "Valuing credit derivatives, *Journal of Fixed Income*, 5, 6-12.
- Longstaff, F., and E. Schwartz, 2001, "Valuing American Options by Simulation: A Simple Least-Squares Approach", *The Review of Financial Studies*, 14, 113-147
- Longstaff, F., P. Santa-Clara, and E. Schwartz, 2001, "The Relative Valuation of Caps and Swaptions: Theory and Empirical Evidence", forthcoming, *Journal of Finance*
- Madan, D. and H. Unal, 1998, "Pricing the risks of default, *Review of Derivatives Research*, 2, 121-160.
- Merton, R., 1974, "On the pricing of corporate debt: The risk structure of interest rates, *Journal of Finance*, 29, 449-470.
- Moraleda J. and T. Vorst, 1997, "Pricing American Interest Rate Claims with Humped Volatility Models," *Journal of Banking and Finance*, 21, 1131-1137.
- Ritchken, P. and I. Chuang, 1999, "Interest Rate Option Pricing with Volatility Humps", *Review of Derivatives Research* 3, 237-262.
- Sarig, O. and A. Warga, 1989, "Some empirical estimates of the risk structure of interest rates, *Journal of Finance*, 44, 1351-1360.
- Schonbucher, P., 2000, "The Pricing of Credit Risk and Credit Risk Derivatives, working paper, Department of Statistics, University of Bonn.

## Appendix

### Proof of Proposition 1

Substituting equations (2) and (3) into equation (1), the forward rate can be written as

$$\begin{aligned} f(t, T) &= f(t-1, T) + \sum_{n=1}^2 \mu_{fn}(t-1, T) \Delta t + \sum_{n=1}^2 \sigma_{fn}(t-1, T) \sqrt{\Delta t} Z_t^{(n)} \\ &= f(0, T) + \sum_{n=1}^2 \sum_{j=0}^{t-1} \mu_{fn}(j, T) \Delta t + \sum_{n=1}^2 \sum_{j=0}^{t-1} \sigma_{fn}(j, T) \sqrt{\Delta t} Z_{j+1}^{(n)} \end{aligned}$$

where  $\mu_{fn}(t, T)$ ,  $n = 1, 2$  is the drift term for each factor, given in equation (4).

Substituting the volatility functions in equations (2) and (3) into the drift term, and defining  $a_1 = e^{-\kappa_1 \Delta t}$ ,  $a_2 = e^{-\kappa_2 \Delta t}$ , we have, upon simplification:

$$\mu_{f1}(t, T) = \frac{b_1^2 a_1^{T-t} \Delta t}{2(a_1 - 1)} (a_1^{T-t} + a_1^{T-t+1} - 2a_1) \quad (34)$$

$$\begin{aligned} \mu_{f2}(t, T) &= \frac{b_2^2 a_1^{T-t+1} \Delta t}{1 - a_1} - \frac{1 + a_1}{2(1 - a_1)} b_2^2 a_1^{2(T-t)} \Delta t \\ &\quad + \frac{c_2^2 a_2^{T-t+1} \Delta t}{1 - a_2} - \frac{1 + a_2}{2(1 - a_2)} c_2^2 a_2^{2(T-t)} \Delta t \\ &\quad + \frac{c_2 b_2 ((1 - a_2) a_2^{T-t} a_1 + (1 - a_1) a_1^{T-t} a_2 + (a_1 a_2 - 1)(a_1 a_2)^{T-t}) \Delta t}{(1 - a_1)(1 - a_2)} \end{aligned} \quad (35)$$

From the above calculation we see that the drift term  $\mu_f(t, T)$  of forward rate is a deterministic function of  $t$  and  $T$ . Let  $h(t, T)$  represent this function. That is:

$$h(t, T) = \sum_{n=1}^2 \sum_{j=0}^t \mu_{fn}(j, T) \Delta t \quad (36)$$

Then:

$$\begin{aligned} f(t, T) &= f(0, T) + \sum_{n=1}^2 \sum_{j=0}^{t-1} \mu_{fn}(j, T) \Delta t + \sum_{n=1}^2 \sum_{j=0}^{t-1} \sigma_{fn}(j, T) \sqrt{\Delta t} Z_{j+1}^{(n)} \\ &= f(0, T) + h(t-1, T) + b_1 a_1^{T-t} \Phi_1(t) + b_2 a_1^{T-t} \Phi_2(t) + c_2 a_2^{T-t} \Phi_3(t) \end{aligned}$$

where the  $\Phi$  terms are defined in the paper.

For  $T = t$ , we have

$$r(t) = f(0, t) + h(t-1, t) + b_1 \Phi_1(t) + b_2 \Phi_2(t) + c_2 \Phi_3(t) \quad (37)$$

Further, we also have

$$r(t+1) = f(0, t+1) + h(t, t+1) + b_1 \Phi_1(t+1) + b_2 \Phi_2(t+1) + c_2 \Phi_3(t+1) \quad (38)$$

Notice

$$\begin{aligned}\Phi_1(t+1) &= a_1\Phi_1(t) + a_1\sqrt{\Delta t}Z_{t+1}^{(1)} \\ \Phi_2(t+1) &= a_1\Phi_2(t) + a_1\sqrt{\Delta t}Z_{t+1}^{(2)} \\ \Phi_3(t+1) &= a_2\Phi_3(t) + a_2\sqrt{\Delta t}Z_{t+1}^{(2)}.\end{aligned}$$

Substituting these terms into the above equation we obtain

$$\begin{aligned}r(t+1) &= f(0, t+1) + h(t, t+1) + a_1(r(t) - f(0, t) - h(t-1, t)) \\ &\quad + (a_2 - a_1)c_2\Phi_3(t) + b_1a_1\sqrt{\Delta t}Z_{t+1}^{(1)} + b_2a_1\sqrt{\Delta t}Z_{t+1}^{(2)} + c_2a_2\sqrt{\Delta t}Z_{t+1}^{(2)}\end{aligned}\quad (39)$$

Let  $\ell(t) = h(t-1, t)$ . The exact expression for  $\ell(t)$  can be obtained by substituting in equations (34) and (35) into equation (36) for  $T = t$  to obtain:

$$\begin{aligned}\ell(t) &= \sum_{j=0}^{t-1} [\mu_{f1}(j, t) + \mu_{f2}(j, t)] \Delta t \\ &= \frac{b_1^2 a_1^2 (a_1^{2t} - 2a_1^t + 1) \Delta t^2}{2(a_1 - 1)^2} + \frac{1}{2} \left[ \frac{b_2 a_1 (a_1^t - 1) \Delta t}{(a_1 - 1)} + \frac{c_2 a_2 (a_2^t - 1) \Delta t}{(a_2 - 1)} \right]^2\end{aligned}\quad (40)$$

The dynamics of the variables  $r(t)$  and  $u(t)$  then follow by transforming the state variable  $\phi_3(t)$ , to  $u(t)$  using  $\phi_3(t) = ku(t)$ .

(ii) These two state variables also determine the term structure. Reconsider the forward rate equation, it can be written as:

$$\begin{aligned}f(t, T) &= f(0, T) + h(t-1, T) + a_1^{T-t}(b_1\Phi_1(t) + b_2\Phi_2(t)) + c_2a_2^{T-t}\Phi_3(t) \\ &= f(0, T) + h(t-1, T) + a_1^{T-t}(r(t) - f(0, t) - h(t-1, t)) + c_2(a_2^{T-t} - a_1^{T-t})ku(t)\end{aligned}$$

(iii) The riskless discount bond price at time  $t$  can be obtained by induction. For  $n = 1$  the bond price equation leads to  $P(t, t+1) = e^{-r(t)\Delta t}$  which is the correct result. Now assume that:

$$P(t, t+n) = e^{-A_n(t) - B_n r(t)\Delta t - C_n u(t)\Delta t}$$

Then,

$$\begin{aligned}P(t, t+n+1) &= e^{-r(t)\Delta t} E[P(t+1, t+n+1)] \\ &= e^{-r(t)\Delta t} E[e^{-A_n(t+1) - B_n r(t+1)\Delta t - C_n u(t+1)\Delta t}]\end{aligned}$$

After computing the expectation, we obtain the result.

### Proof of Proposition 2

Recall that the state variables for interest rate can be written as:

$$\begin{aligned}
\Delta r(t) &= \left[ \frac{f(0, t+1) - f(0, t)}{\Delta t} \right] \Delta t + \left[ \frac{\ell(t+1) - \ell(t)}{\Delta t} \right] \Delta t \\
&\quad + (a_1 - 1)[r(t) - f(0, t) - \ell(t-1)] + c_2(a_2 - a_1)ku(t) + d_1\sqrt{\Delta t}\epsilon_{t+1} \\
\Delta u(t) &= (a_2 - 1)u(t) + a_2/k\sqrt{\Delta t}\eta_{t+1}
\end{aligned}$$

We now rewrite the above equation, where the arguments are all in calendar time rather than in periods. Specifically, over the interval  $[t, t + \Delta t]$  we obtain:

$$\begin{aligned}
\Delta r(t) &= \left[ \frac{f(0, t + \Delta t) - f(0, t)}{\Delta t} \right] \Delta t + \left[ \frac{\ell(t + \Delta t) - \ell(t)}{\Delta t} \right] \Delta t \\
&\quad + (a_1 - 1)[r(t) - f(0, t) - \ell(t - \Delta t)] + c_2(a_2 - a_1)ku(t) + d_1\sqrt{\Delta t}\epsilon_{t+\Delta t} \quad (41)
\end{aligned}$$

$$\Delta u(t) = (a_2 - 1)u(t) + a_2k\sqrt{\Delta t}\eta_{t+\Delta t} \quad (42)$$

Now as  $\Delta t \rightarrow 0$ , the first term converges to  $f'(0, t)dt$ , and the second term converges to  $\bar{u}'(0, t)dt$ , say, where  $\bar{u}(0, t) = \lim_{\Delta t \rightarrow 0} \ell(t)$ , where  $\ell(t)$  is given by equation (40). Substituting for  $a_1$  and  $a_2$  and taking the limits as  $\Delta t \rightarrow 0$  leads to equation (14). The remaining terms in the second row of equation (41) converges to:

$$-\kappa_1[r(t) - f(0, t) - \bar{u}(t)]dt + c_2(\kappa_1 - \kappa_2)ku(t)dt + \sigma_1dw(t)$$

where

$$\sigma_1 = \lim_{\Delta t \rightarrow 0} d_1 = \sqrt{b_1^2 + (b_2 + c_2)^2}.$$

Now with  $k = \frac{\kappa_1}{(\kappa_1 - \kappa_2)c_2}$  the above expression reduces to:

$$\kappa_1[f(0, t) - r(t) + \bar{u}(t) + u(t)]dt + \sigma_1dw(t).$$

Finally, the dynamics of the auxiliary state variable  $u(t)$  as given in equation (42) can easily be shown to converge to equation (28).

### Proof of Proposition 3

The results can be obtained by induction. When  $n = 1$ ,  $G(t, t + 1) = e^{-r(t)\Delta t - s(t)\Delta t}$ , which is correct. Assume  $G(t, t + n) = e^{-\bar{A}_n(t) - \bar{B}_nr(t)\Delta t - \bar{C}_nu(t)\Delta t - \bar{D}_ns(t)\Delta t}$  Then

$$\begin{aligned}
G(t, t + n + 1) &= e^{-r(t)\Delta t - s(t)\Delta t} E[G(t + 1, t + n + 1)] \\
&= e^{-r(t)\Delta t - s(t)\Delta t} E[e^{-\bar{A}_n(t+1) - \bar{B}_nr(t+1)\Delta t - \bar{C}_nu(t+1)\Delta t - \bar{D}_ns(t+1)\Delta t}] \\
&= e^{-(1 + \bar{B}_na_1)r(t)\Delta t - (1 + \bar{D}_n\alpha_1)s(t)\Delta t - (\bar{B}_nc_1 + \bar{C}_na_2)u(t)\Delta t} \\
&\quad \cdot e^{-\bar{A}_n(t+1) - \bar{B}_n(f(0, t+1) + l(t+1) - a_1(f(0, t) + l(t)))\Delta t - \bar{D}_n(\alpha_0 + \alpha_2\alpha_3^2)\Delta t} \\
&\quad \cdot E[e^{-\bar{B}_nd_1(\Delta t)^{3/2}\epsilon_{t+1} - \bar{C}_nd_2(\Delta t)^{3/2}\eta_{t+1} - \bar{D}_n\alpha_2\Delta t(\zeta_{t+1}^2 - 2\alpha_3\zeta_{t+1})}]
\end{aligned}$$

Let

$$\begin{aligned}\eta_{t+1} &= \rho_3 \zeta_{t+1} + \sqrt{1 - \rho_3^2} \bar{\eta}_{t+1}, \quad E[\zeta_{t+1} \bar{\eta}_{t+1}] = 0 \\ \epsilon_{t+1} &= \rho_2 \zeta_{t+1} + \sqrt{1 - \rho_2^2} \bar{\epsilon}_{t+1}, \quad E[\zeta_{t+1} \bar{\epsilon}_{t+1}] = 0\end{aligned}$$

Since  $E[\eta_{t+1} \epsilon_{t+1}] = \rho_1$ ,  $E[\bar{\eta}_{t+1} \bar{\epsilon}_{t+1}] = \frac{\rho_1 - \rho_3 \rho_2}{\sqrt{(1 - \rho_3^2)(1 - \rho_2^2)}}$ . Denote  $\rho_4 = E[\bar{\eta}_{t+1} \bar{\epsilon}_{t+1}]$ . We can write  $\bar{\eta}_{t+1} = \rho_4 \bar{\epsilon}_{t+1} + \sqrt{1 - \rho_4^2} \bar{\bar{\eta}}_{t+1}$ ,  $E[\bar{\zeta}_{t+1} \bar{\bar{\eta}}_{t+1}] = 0$ . Since  $E[\zeta_{t+1} \bar{\eta}_{t+1}] = 0$  and  $E[\zeta_{t+1} \bar{\epsilon}_{t+1}] = 0$  we have  $E[\zeta_{t+1} \bar{\bar{\eta}}_{t+1}] = 0$ . Now we can calculate the expectation using three uncorrelated variables,  $\zeta_{t+1}$ ,  $\bar{\epsilon}_{t+1}$ , and  $\bar{\bar{\eta}}_{t+1}$ . Specifically,

$$\begin{aligned}& E[e^{-\bar{B}_n d_1 \epsilon_{t+1} (\Delta t)^{\frac{3}{2}} - \bar{C}_n d_2 \eta_{t+1} (\Delta t)^{\frac{3}{2}} - \bar{D}_n \alpha_2 \Delta t (\zeta_{t+1}^2 - 2\alpha_3 \zeta_{t+1})}] \\ &= E[e^{-\bar{B}_n d_1 (\Delta t)^{\frac{3}{2}} (\rho_2 \zeta_{t+1} + \sqrt{1 - \rho_2^2} \bar{\epsilon}_{t+1}) - \bar{C}_n d_2 (\Delta t)^{\frac{3}{2}} (\rho_3 \zeta_{t+1} + \sqrt{1 - \rho_3^2} \bar{\eta}_{t+1}) - \bar{D}_n \alpha_2 \Delta t (\zeta_{t+1}^2 - 2\alpha_3 \zeta_{t+1})}] \\ &= E[e^{-(\bar{B}_n d_1 \sqrt{1 - \rho_2^2} + \bar{C}_n d_2 \sqrt{1 - \rho_3^2} \rho_4) \bar{\epsilon}_{t+1} (\Delta t)^{\frac{3}{2}} - \bar{C}_n d_2 \sqrt{1 - \rho_3^2} \sqrt{1 - \rho_4^2} \bar{\bar{\eta}}_{t+1} (\Delta t)^{\frac{3}{2}} \\ &\quad \cdot e^{-\bar{D}_n \alpha_2 \zeta_{t+1}^2 \Delta t + (2\alpha_3 \alpha_2 \bar{D}_n - (\bar{B}_n d_1 \sqrt{\Delta t} \rho_2 + \bar{C}_n d_2 \sqrt{\Delta t} \rho_3)) \zeta_{t+1} \Delta t}] \\ &= e^{\frac{\Delta t^3}{2} (\bar{B}_n d_1 \sqrt{1 - \rho_2^2} + \bar{C}_n d_2 \sqrt{1 - \rho_3^2} \rho_4)^2 + \frac{\Delta t^3}{2} (\bar{C}_n d_2 \sqrt{1 - \rho_3^2} \sqrt{1 - \rho_4^2})^2} \\ &\quad \cdot e^{-\frac{1}{2} \ln(1 + 2\alpha_2 \bar{D}_n \Delta t) + \frac{(\bar{B}_n d_1 \sqrt{\Delta t} \rho_2 + \bar{C}_n d_2 \sqrt{\Delta t} \rho_3 - 2\alpha_2 \alpha_3 \bar{D}_n)^2 \Delta t^2}{2(1 + 2\alpha_2 \bar{D}_n \Delta t)}}\end{aligned}$$

Eventually, we have

$$G(t, t + n + 1) = e^{-\bar{A}_{n+1}(t) - \bar{B}_{n+1}r(t)\Delta t - \bar{C}_{n+1}u(t)\Delta t - \bar{D}_{n+1}s(t)\Delta t}.$$

Comparing  $P(t, t + n)$  and  $G(t, t + n)$ , we see that  $B_n = \bar{B}_n$  and  $C_n = \bar{C}_n$ . Hence we can write  $G(t, t + n) = P(t, t + n) e^{-\bar{D}_n s(t) \Delta t - \bar{E}_n(t)}$ .

Actually,

$$\bar{E}_n(t) = \bar{A}_n(t) - A_n(t) = \bar{E}_n.$$

The time-varying parts cancel out in  $\bar{E}_n(t)$ . Therefore,

$$G(t, t + n) = P(t, t + n) e^{-\bar{D}_n s(t) \Delta t - \bar{E}_n}$$

#### Proof of Proposition 4

(i) Consider the diffusion limit when  $r(t)$  and  $s(t)$  use different stochastic drivers,  $r(t)$  and  $u(t)$  converge to the same process discussed before, while  $s(t)$  can be written as

$$\begin{aligned}s(t + 1) - s(t) &= \alpha_0 + (\alpha_1 - 1)s(t) + \alpha_2(\zeta_{t+1} - \alpha_3)^2 \\ &= \alpha_0 + (\alpha_1 - 1)s(t) + \alpha_2(\zeta_{t+1}^2 - 1) - 2\alpha_2\alpha_3\zeta_{t+1} + \alpha_2(1 + \alpha_3^2) \\ &= \bar{\alpha}_0 \Delta t - \bar{\alpha}_1 s(t) \Delta t + \bar{\alpha}_2 \sqrt{\Delta t} (\zeta_{t+1}^2 - 1) - 2\bar{\alpha}_2 \bar{\alpha}_3 \zeta_{t+1} \sqrt{\Delta t}\end{aligned}$$

where  $\bar{\alpha}_0 \Delta t = \alpha_0 + \alpha_2(1 + \alpha_3^2)$ ,  $\bar{\alpha}_1 \Delta t = 1 - \alpha_1$ ,  $\bar{\alpha}_2 \sqrt{\Delta t} = \alpha_2$ , and  $\bar{\alpha}_3 = \alpha_3$ . As  $\Delta t$  goes to 0, we have

$$\begin{aligned}
dr(t) &= (f'(0, t) + \bar{u}'(0, t) + \kappa_1(u(t) - r(t) + f(0, t) + \bar{u}(0, t))dt + \sigma_1 dw_1 \\
du(t) &= -\kappa_2 u(t)dt + \sigma_2 dw_2(t) \\
ds(t) &= (\bar{\alpha}_0 - \bar{\alpha}_1 s(t))dt + \sqrt{2}\bar{\alpha}_2 dw_3(t) - 2\bar{\alpha}_2 \bar{\alpha}_3 dw_4(t) \\
\bar{u}(0, t) &= \frac{b_1^2(1 - e^{-\kappa_1 t})^2}{2\kappa_1^2} + \frac{1}{2} \left[ \frac{b_2(1 - e^{-\kappa_1 t})}{\kappa_1} + \frac{c_2(1 - e^{-\kappa_2 t})}{\kappa_2} \right]^2 \\
Cov[dw_1(t)dw_2(t)] &= \rho dt \\
Cov[dw_1(t)dw_3(t)] &= Cov[\epsilon_{t+1}, \zeta_{t+1}^2]dt = 2E[\zeta_{t+1}]Cov[\epsilon_{t+1}, \zeta_{t+1}]dt = 0 \\
Cov[dw_1(t)dw_4(t)] &= (\rho q_2 + \sqrt{1 - \rho^2} q_1)dt \\
Cov[dw_2(t)dw_3(t)] &= Cov[\eta_{t+1}, \zeta_{t+1}^2]dt = 2E[\zeta_{t+1}]Cov[\eta_{t+1}, \zeta_{t+1}]dt = 0 \\
Cov[dw_2(t)dw_4(t)] &= q_2 dt
\end{aligned}$$

(ii) Assume  $G^c(t, T) = e^{-\theta(t, T) - \beta(t, T)r(t) - \gamma(t, T)u(t) - \xi(t, T)s(t)}$

The differential equation satisfied by the discount risky bond price is

$$\begin{aligned}
G_t^c + G_r^c dr + G_u^c du + G_s^c ds + \frac{1}{2}\sigma_2^2 G_{uu}^c + \frac{1}{2}\bar{\alpha}_2^2(2 + 4\bar{\alpha}_3^2)G_{ss}^c \\
+ \sigma_1 \sigma_2 \rho G_{ru}^c - 2\sigma_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\rho}_2 G_{sr}^c - 2\sigma_2 \bar{\alpha}_2 \bar{\alpha}_3 \rho_3 G_{us}^c = G^c(r + s)
\end{aligned}$$

That is:

$$\begin{aligned}
-\theta' - \beta' r - \gamma' u - \theta' s - \beta(f'(0, t) + \bar{u}'(0, t) + \kappa_1(u - r + f(0, t) + \bar{u}(0, t))) - \gamma(-\kappa_2 u) \\
-\xi(\bar{\alpha}_0 - \bar{\alpha}_1 s) + \sigma_1 \sigma_2 \rho \beta \gamma - 2\sigma_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\rho}_2 \xi \beta - 2\sigma_2 \bar{\alpha}_2 \bar{\alpha}_3 \rho_3 \xi \gamma \\
+ \frac{1}{2}\sigma_1^2 \beta^2 + \frac{1}{2}\sigma_2^2 \gamma^2 + \frac{1}{2}\bar{\alpha}_2^2(2 + 4\bar{\alpha}_3^2)\theta^2 - r - s = 0.
\end{aligned}$$

Then:

$$\begin{aligned}
-\beta' r + \beta \kappa_1 r - r &= 0 \\
-\gamma' u - \beta \kappa_1 u + \gamma \kappa_2 u &= 0 \\
-\xi' s - \xi \bar{\alpha}_1 s - s &= 0 \\
-\alpha' - \beta(f'(0, t) + \bar{u}'(0, t) + \kappa_1(f(0, t) + \bar{u}(0, t))) - \xi \bar{\alpha}_0 \\
+ \sigma_1 \sigma_2 \rho \beta \gamma - 2\sigma_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\rho}_2 \xi \beta - 2\sigma_2 \bar{\alpha}_2 \bar{\alpha}_3 \rho_3 \xi \gamma \\
+ \frac{1}{2}\sigma_1^2 \beta^2 + \frac{1}{2}\sigma_2^2 \gamma^2 + \frac{1}{2}\bar{\alpha}_2^2(2 + 4\bar{\alpha}_3^2)\theta^2 &= 0
\end{aligned}$$

Satisfying boundary conditions,  $\beta(t, t) = 0$ ,  $\gamma(t, t) = 0$ ,  $\xi(t, t) = 0$  and  $\theta(t, t) = 0$ . Solving the differential equation gives the result.

### Proof of Proposition 5

We use induction to prove the results.

$$D_0(t, t+1) = E_0[e^{-r(t)\Delta t - \sum_{j=0}^{t-1} (r(j)+\lambda(j))\Delta t}] - E_0[e^{-\sum_{j=0}^t (r(j)+\lambda(j))\Delta t}] = A_t - G^0(0, t+1) \quad (43)$$

When  $t = 1$ ,

$$\begin{aligned} A_1 &= E_0[-e^{r(1)\Delta t - (r(0)+\lambda(0))\Delta t}] \\ &= e^{-(r(0)+\lambda(0))\Delta t} E_0[e^{-(f(0,1)+l(1)+a_1(r(0)-f(0,0)-l(0))+c_1u(0)+d_1\sqrt{\Delta t}\epsilon_1)\Delta t}] \\ &= e^{-X_1r(0)\Delta t - Y_1\lambda(0)\Delta t - W_1u(0)\Delta t - M(1,1)} \end{aligned}$$

where

$$\begin{aligned} X_1 &= 1 + a_1 \\ Y_1 &= 1 \\ W_t &= c_1 \\ M(1,1) &= [f(0,1) + \ell(1) - a_1(f(0,0) - \ell(0))]\Delta t - \frac{d_1^2(\Delta t)^3}{2} \\ M(1,t) &= [f(0,t) + \ell(t) - a_1(f(0,t-1) - \ell(t-1))]\Delta t - \frac{d_1^2(\Delta t)^3}{2} \end{aligned}$$

Assume under  $t - 1$

$$\begin{aligned} A_{t-1} &= E_0[-e^{r(t-1)\Delta t - \sum_{j=0}^{t-2} (r(j)+\lambda(j))\Delta t}] \\ &= e^{-X_{t-1}r(0)\Delta t - Y_{t-1}\lambda(0)\Delta t - W_{t-1}u(0)\Delta t - M(t-1, t-1)} \end{aligned}$$

where

$$\begin{aligned} X_{t-1} &= 1 + a_1X_{t-2}, \\ Y_{t-1} &= 1 + \alpha_1Y_{t-2}, \\ W_{t-1} &= c_1X_{t-2} + a_2W_{t-2}, \\ M(t-1, t-1) &= M(t-2, t-1) + X_{t-1}[f(0,1) + \ell(1) - a_1(f(0,0) - \ell(0))]\Delta t + Y_{t-1}(\alpha_0 + \alpha_2\alpha_3^3)\Delta t \\ &\quad + \frac{1}{2}\ln(1 + 2Y_{t-2}\alpha_2\Delta t) - \frac{(X_{t-2}d_1\sqrt{\Delta t} + W_{t-2}d_2\rho_1 - 2Y_{t-2}\alpha_2\alpha_3)^2(\Delta t)^2}{2(1 + 2Y_{t-2}\alpha_2\Delta t)} \\ &\quad - \frac{1}{2}W_{t-2}^2(1 - \rho_1^2)d_2^2(\Delta t)^3 \end{aligned}$$

Then at time  $t$ ,

$$\begin{aligned} A_t &= E_0[e^{-r(t)\Delta t - \sum_{j=0}^{t-1} (r(j)+\lambda(j))\Delta t}] \\ &= E_0[e^{-(r(0)+\lambda(0))\Delta t} E_1[e^{-r(t)\Delta t - \sum_{j=1}^{t-1} (r(j)+\lambda(j))\Delta t} | F_1]] \\ &= E_0[e^{-(r(0)+\lambda(0))\Delta t} e^{-X_{t-1}r(1)\Delta t - Y_{t-1}\lambda(1)\Delta t - W_{t-1}u(1)\Delta t - M(t-1, t)}] \end{aligned}$$

After simplifying the above expectation, we get the result.

Table 1: Percentage Errors(%) in the Estimated Prices of Swaptions\*

Swaption Expiration	Swap Maturity				
	1	2	3	4	5
0.5	0.9162	2.7156	2.0826	0.2835	-1.8188
1	-2.7209	0.6341	1.6773	1.987	0.9834
2	-4.1604	1.9576	2.3977	2.0118	1.0799
3	-2.9639	2.3639	2.7782	3.0003	3.0625
4	-5.9532	-1.0083	-0.8853	-0.1778	0.7692
5	-7.1496	-3.5083	-2.0218	-0.3679	-0.1448

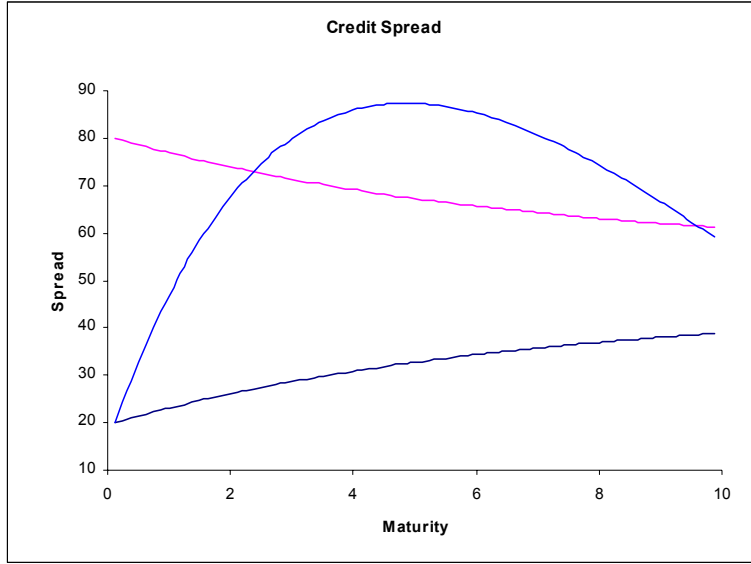
\* Table 1 shows the percentage errors in swaption prices for September 30<sup>th</sup>, 1999. The optimization procedure is described in the text.

Table 2: Price Information on Subordinated Bonds Issued by Chase (BHC) \*

Coupon	Maturity	1st Coupon	Price	Theoretical Price	Percentage Error (%)
6	2/15/09	8/15/99	92.16	91.77	-0.4055
6.375	2/15/08	8/15/98	95.51	95.02	-0.5157
6.375	4/1/08	10/1/98	95.51	97.26	1.8276
7.125	2/1/07	8/1/97	99.98	99.75	-0.2300
7.125	6/15/09	12/15/99	99.98	99.20	-0.7763
7.25	6/1/07	12/1/97	101.1	101.36	0.2616
7.5	2/1/03	8/1/93	102.03	101.99	-0.0356
8.625	5/1/02	11/1/92	104.36	104.60	0.2296

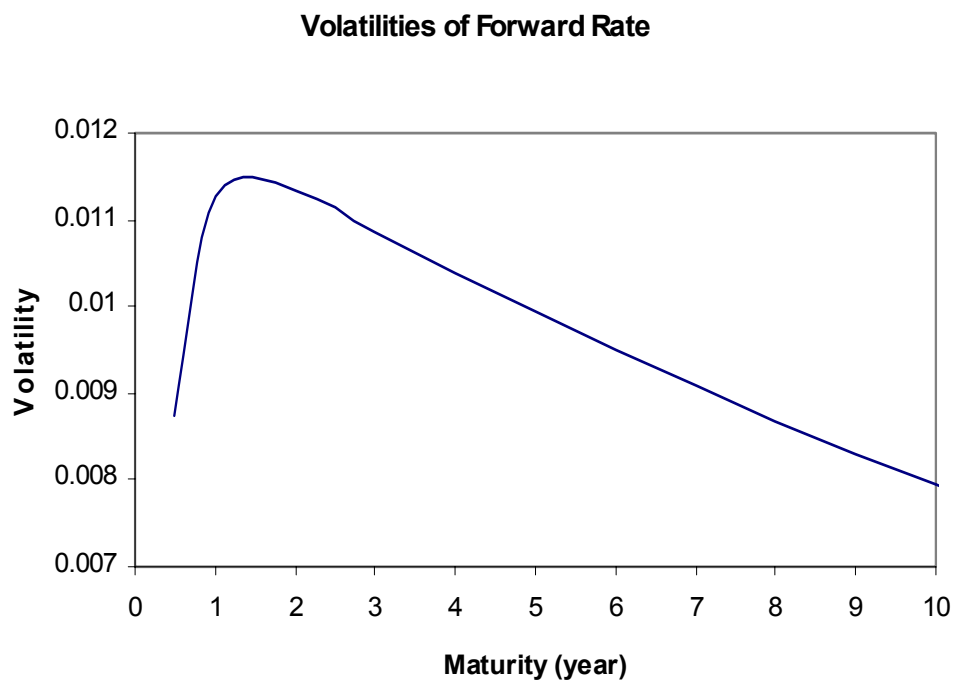
\*Table 2 shows the actual prices of selected Chase Bank Holding Company subordinated bond issues on September 30<sup>th</sup>, 1999. Also shown are the estimated bond prices and the percentage errors in prices of the bonds.

Figure 1: Different Shape of the Term Structure of Credit Spreads\*



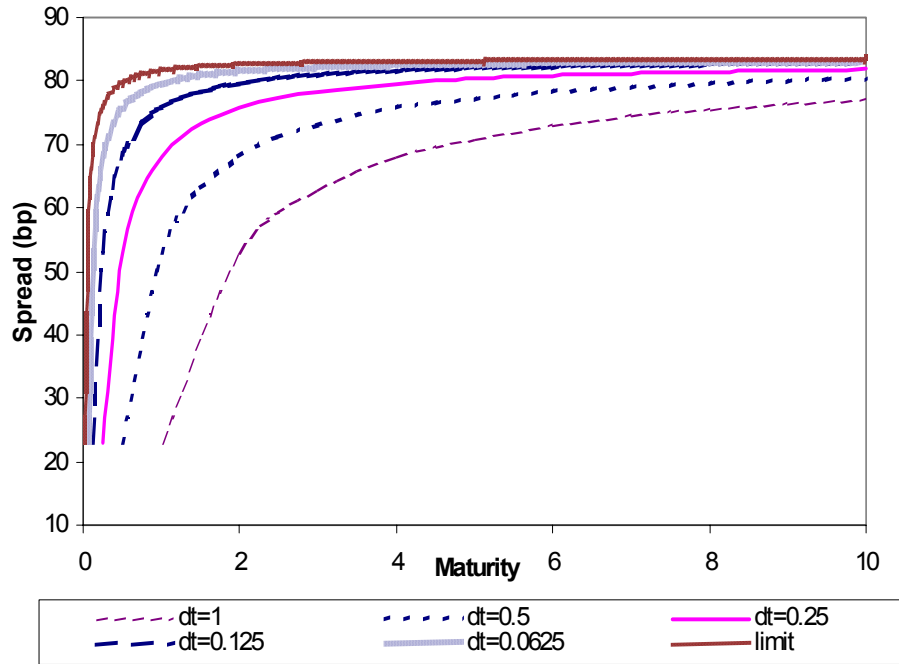
The figure shows different shapes of the credit spread curve that can be established by the model. The benchmark case parameters for the interest rate process are  $\kappa_1 = 0.044978$ ;  $\kappa_2 = 3.407608$ ;  $b_1 = 0.00014383$ ;  $b_2 = -0.012441$ ;  $c_2 = 0.018797$ ;  $\rho_2 = 1$ ;  $\rho_3 = 0.9997$ . When,  $\alpha_0 = 0.003199$ ;  $\alpha_1 = 0.00008$ ;  $\alpha_2 = 0.001924$ ;  $\alpha_3 = 1.304466$ ;  $s(0) = 0.002$ , the term structure is upward sloping. With the same parameters, but  $s(0) = 0.009$ , the curve is downward sloping. Finally, when  $\alpha_0 = -0.013229167$ ;  $\alpha_1 = 0.98125$ ;  $\alpha_2 = 0.0075$ ;  $\alpha_3 = 0.942809042$ ; and  $s(0) = 0.002$ , the term structure has a mild hump shape.

Figure 2: Estimated Volatility Hump of Forward Rates\*



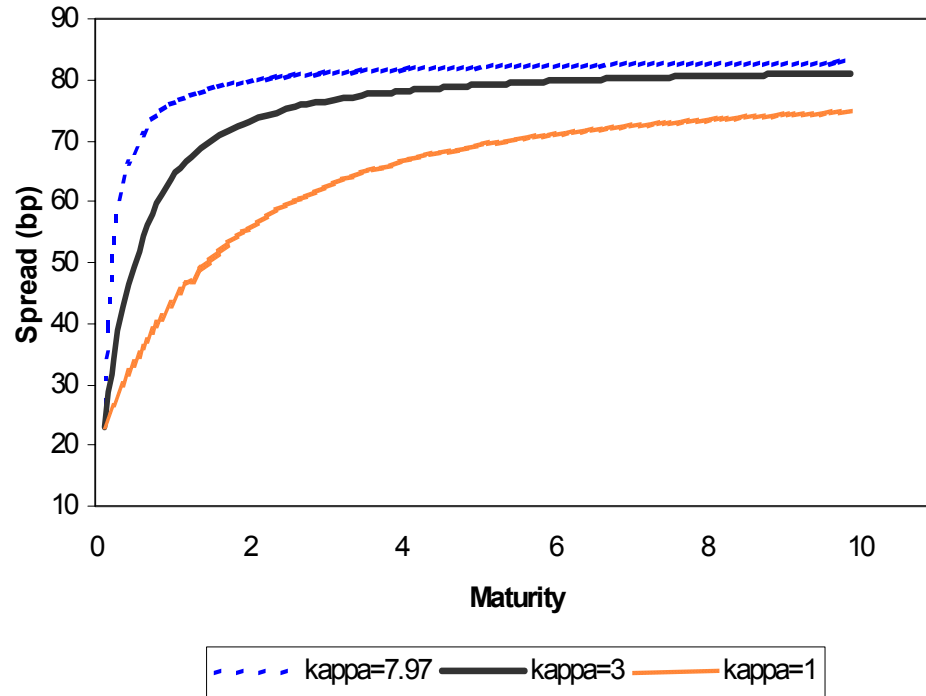
\* The volatility structure of forward rates was estimated using data on the LIBOR term structure and concurrent swaption prices as discussed. The estimated parameters are  $\kappa_1 = 0.044978$ ;  $\kappa_2 = 3.407608$ ;  $b_1 = 0.00014383$ ;  $b_2 = -0.012441$ ;  $c_2 = 0.018797$ .

Figure 3: Term Structure of Credit Spreads from Different Discrete Time Models\*



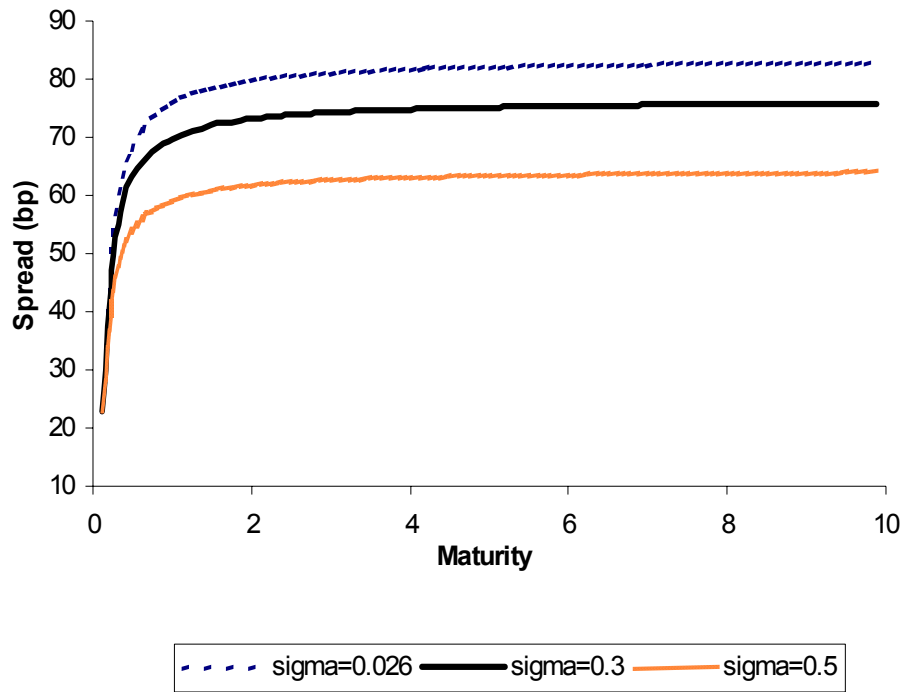
\* The figure shows the convergence rate of the credit spread of discrete time models to their diffusion limit as the time increment is refined. In particular, the credit spreads are shown for  $\Delta t = 1, 0.5, 0.25, 0.125, 0.0625$  and for the diffusion limit. The parameter values used here are;  $\kappa_1 = 0.044978$ ,  $\kappa_2 = 3.407608$ ,  $b_1 = 0.00014383$ ,  $b_2 = -0.012441$ ,  $c_2 = 0.018797$ ,  $\rho_2 = 0.1$ ,  $\rho_3 = 0.3$ ,  $s(0) = 0.0023$ ,  $\alpha_0 = 0.001814$ ,  $\alpha_1 = 0.003571$ ,  $\alpha_2 = 0.0065$ , and  $\alpha_3 = -0.000427$ .

Figure 4: The Effects of Mean Reversion on Credit Spreads\*



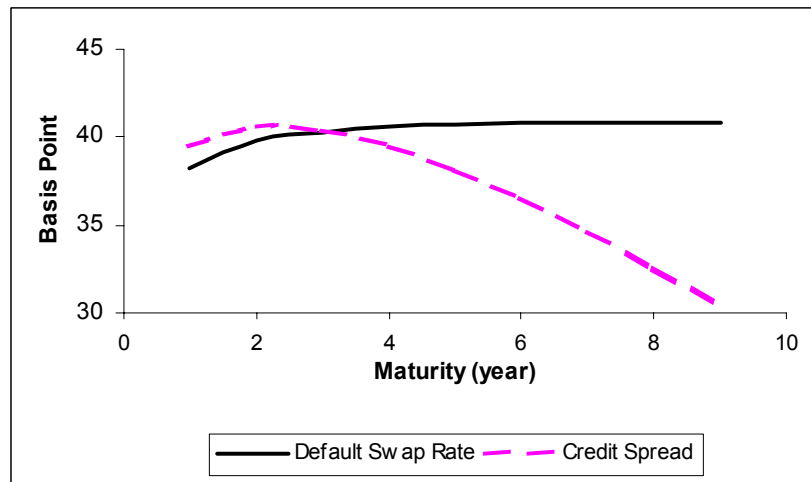
The figure shows the effect of different mean reversion parameter values on the shape of the term structure of credit spreads. We have  $\kappa = \bar{\alpha}_1$ . Here  $\kappa_1 = 0.044978$ ;  $\kappa_2 = 3.407608$ ;  $b_1 = 0.00014383$ ;  $b_2 = -0.012441$ ;  $c_2 = 0.018797$ ;  $s(0) = 0.0023$ ;  $\frac{\bar{\alpha}_0}{\bar{\alpha}_1} = 0.0083$ ;  $\bar{\alpha}_2 = 0.018385$ , and  $\bar{\alpha}_3 = -0.000427$ .

Figure 5: The Effects of Volatility on the Credit Term Structure\*



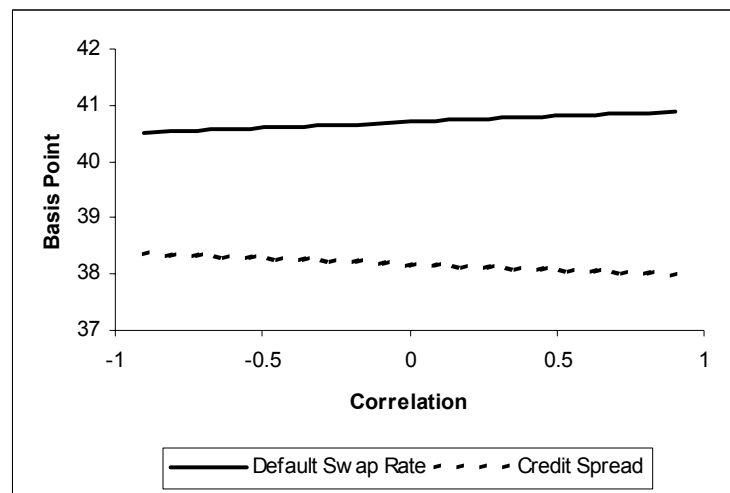
\*The figure shows the effects of increasing the volatility of the short spread on the term structure of credit spreads. Here,  $\sigma = \bar{\alpha}_2 \sqrt{2 + 4\bar{\alpha}_3^2}$ . The values are:  $\kappa_1 = 0.044978$ ;  $\kappa_2 = 3.407608$ ;  $b_1 = 0.00014383$ ;  $b_2 = -0.012441$ ;  $c_2 = 0.018797$ ;  $s(0) = 0.0023$ ;  $\bar{\alpha}_0 = 0.06652$ ;  $\bar{\alpha}_1 = 7.97$ ; and  $\bar{\alpha}_3 = -0.00427$ .

Figure 6: The Effects of Maturity on the Credit and Default Swap Rate Curves\*



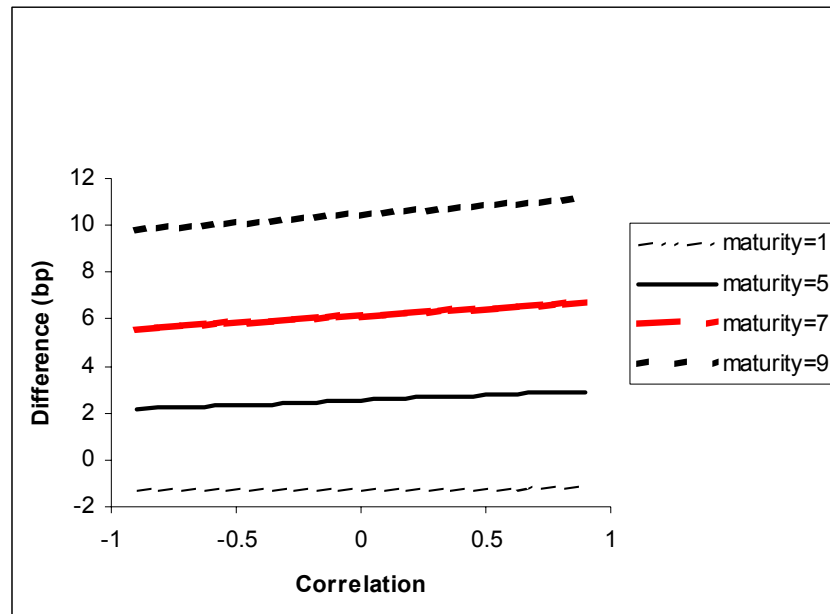
\* The figure shows the default swap rate curve and the credit spread curve when the initial term structure of interest rates are flat at 5%. In this figure the correlation between the spread and interest rate is 0 and the initial term structure is flat at 5%. The parameters used in the analysis were those estimated from Chase Bank, except for the correlation which was set at 0.

Figure 7: The Effects of Correlation on the Credit and Default Swap Rate Curves\*



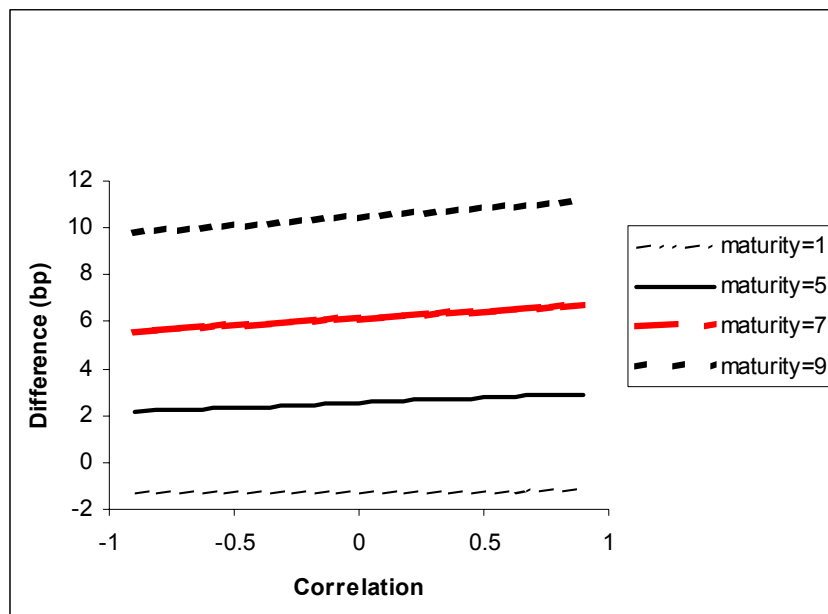
\* The figure shows the default swap rate curve and the credit spread curve for 5 year contracts when the initial term structure of interest rates are flat at 5%. The parameters used in the analysis correspond to the estimated values used in the previous figures. As the correlation was changed, the long run mean and standard deviation of the short credit spread were kept constant at their calibrated values of 0.08345, and 0.026 respectively.

Figure 8: Spread Between Default Swap Rates and Credit Spreads\*



\* The figure shows the effects of increasing correlation on the credit spreads and default swap rates, when the initial term structure of interest rates are flat at 5%. The parameters used in the analysis correspond to the estimated parameters used in the previous figures. As the correlation was changed, the long run mean and standard deviation of the credit spread were kept constant at their calibrated values of 0.08345, and 0.026 respectively.

Figure 9: Percentage Differences Between Default Swap and Credit Spreads\*



\* The figure shows the differences between the default swap rate and the credit spread expressed as a percentage of the swap rate. The data used in the analysis correspond to the data in the previous figure.