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The Importance of Forward-Rate Volatility Structures in Pricing Interest Rate-Sensitive Claims*

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Studies of the sensitivity of the prices of interest rate claims to alternative specifications of the volatility of spot and forward interest rates have drawn different conclusions. One possible explanation for this is that it is difficult to adjust the volatility structure without disturbing the initial set of bond prices. In this chapter we use a term structure-constrained model that lets us change the volatility structure for spot and forward rates without altering either their initial values or the set of initial bond prices. Consequently, any differences in prices of interest rate-sensitive claims can be attributed solely to alternative assumptions on the structure *of spot and forward-rate volatilities rather than to variations in the initial conditions. We show that even when the initial conditions are common, option prices on interest rates and on bonds are sensitive to the specification of the volatility structure of spot rates. Further, we find that using a simple generalised Vasicek model to price claims can lead to significant mispricings if interest rate volatilities do indeed depend on their levels.*

In the last decade, over-the-counter trading in interest rate derivatives has expanded dramatically to a multi-trillion dollar market of notional principal. The majority of claims in this market are now quoted and priced relative to an existing term structure. As a result, recent research has attempted to represent the co-movements of all bond prices so that information from the existing term structure is more fully reflected in model values.

A key ingredient of all models of the term structure is the choice of a volatility structure for spot and forward interest rates. Our goal is to investigate the importance of

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different specifications of spot and forward-rate volatility structures on the pricing of interest rate-sensitive securities. We examine in particular the effect that the volatility specification has on the pricing of short- and long-term bond and yield options.

The empirical evidence on the importance of volatility structures is mixed. Hull and White (1990) introduced time-varying parameters into the Vasicek (1977) and the Cox, Ingersoll and Ross (1985) models so that the level of the term structure and its volatility could be initialised to exogenously given values. Using simulation techniques, they found that the prices of short-dated options produced by the two models are similar. This being the case, they argued that a simple Vasicek-type volatility structure could serve as an acceptable proxy in computing option prices even if the true volatility structure is of the square root form.

In a different study, Chan *et al* (1992) provided empirical evidence that the volatility of the spot interest rate is quite sensitive to its level and that option prices are sensitive to alternative volatility specifications. In the models they tested, however, the initial prices of discount bonds change with the volatility specification for the spot interest rate.[1] As a result, the sensitivity of option prices is not attributable solely to the different specifications of spot rate volatility but also arises because the underlying bond prices change.

In this chapter we focus on the pricing of interest rate-sensitive claims when the models are initialised to the *same* term structure and to the *same* initial set of forward-rate volatilities. Consequently, differences in option prices produced by our models arise solely from different assumptions placed on the *structure* for the spot and forward-rate volatilities.[2] If many realistic structures for forward-rate volatilities produce prices close to those produced by the simple generalised Vasicek structure, then, as suggested by Hull and White, such a model may well serve as a benchmark model. These models, which are characterised by deterministic volatility structures, have the advantage of yielding analytical solutions for many claims and are very tractable. If, on the other hand, we find that option prices are very sensitive to the volatility specification, this suggests that more attention should be given to estimating the true volatility structure for forward rates.

Our results show that the use of generalised Vasicek volatility structure models as a proxy when the true volatilities are not of this type can lead to significant mispricings. We need more empirical studies to identify a viable volatility structure of forward rates.

Volatility structures and the pricing of interest rate claims

Let $P(t, T)$ be the price at date t of a pure discount bond that matures at time T and let $f(t, T)$ be the forward interest rate, viewed from time t, for the time increment $[T, T + dT]$. Here, $f(t, t) = r(t)$ is the spot interest rate for time t. By definition, forward rates and bond prices are related as

$$p(t,T) = e^{-\int_t^T f(t,s)\,ds} \tag{1}$$

Most models of the term structure either explicitly or implicitly specify some dynamics for these forward rates and bond prices. We assume that the evolution of the forward-rate curve can be described by a single-factor diffusion process of the form

$$df(t,T) = \mu_f(t,T)\,dt + \sigma[r(t)]^{\gamma} e^{-\kappa(T-t)}\,dw(t) \qquad \text{for } T > t \tag{2}$$

In this representation the volatility of the spot interest rate at time t is given as

$$\sigma[r(t)]^{\gamma} \tag{3}$$

for some parameters σ and γ.[3]

Many studies of spot interest rate behaviour have adopted this type of structure for spot rate volatilities. For example, in the Vasicek (1977) model, $\gamma = 0$. The celebrated Cox, Ingersoll and Ross (1985) model sets $\gamma = 0.5$, while Dothan (1978) considered a proportional model where $\gamma = 1.0$. The empirical study of Chan *et al* (1992) suggests that γ may even be as high as $1\frac{1}{2}$. In this study we examine the impact of γ on the pricing of options on bonds and yields.

As seen in Equation (2), we set the volatility of forward rates to decay exponentially with maturity. This "exponentially dampened" representation has been used in many other studies, including Turnbull and Milne (1991), Jamshidian (1989), Heath, Jarrow and Morton (1992) and as far back as Vasicek (1977).[4] If κ is positive, shocks to the term structure have an exponentially dampened effect across maturities. Near-term forward rates will have volatilities "close" to the volatility of the spot rate, while more distant forward rates will be less affected. The structure therefore captures the notion that distant forward rates are less volatile than near-term rates.[5]

Although this volatility structure does not incorporate all possible forms, it does incorporate a very significant subset. In the analysis that follows we restrict attention to these structures. Notice that the form of the volatilities is completely characterised by the selection of three parameters, σ, γ and κ. σ is a scaling parameter, γ is the "elasticity" measure and κ captures the dampening effect of volatilities across the term structure.

Ritchken and Sankarasubramanian (1995) have shown that for this specification of term structure dynamics the forward-rate curve at any future date can be derived in its entirety in a simple manner once any *two* points on that future forward-rate curve are known. This is in contrast to the Vasicek and Cox–Ingersoll–Ross models, where the spot interest rate serves as the sole variable determining the forward-rate curve. The additional burden imposed by this second state variable is offset by an ability to specify any value for the elasticity parameter, γ, and also by being able to initialise the model to any initial term structure without introducing time-dependent parameters that are hard to estimate.

The model follows Heath, Jarrow and Morton (1992). Without restricting the structure of volatilities, however, the forward-rate curve cannot be characterised by a finite set of points and the dynamics on an approximating lattice may be path-dependent. This is discussed in Heath, Jarrow and Morton, Amin and Morton (1994) and Li, Ritchken and Sankarasubramanian (1995). Amin and Morton (1994) tested alternative volatility structures using a non-recombining lattice with a small number of time partitions.

Ritchken and Sankarasubramanian (1995) also showed that the two state variables that describe the shape and evolution of the term structure can be derived as the spot interest rate, $r(T)$, and a second statistic, $\phi(T)$, which describes the accumulated volatility of the spot rate over the interval $(0, T)$. Given the levels of these two variables, the entire term structure can be recovered.

To compare the price of an option, we need to know the terminal distribution of these two state variables under the risk-adjusted process described in the appendix. Unfortunately, when $\gamma \neq 0$, the resulting distribution is non-standard and must be estimated numerically. When $\gamma = 0$, the model reduces to the generalised Vasicek model and analytical solutions are available for almost any interest rate claim. The main pricing results developed by Ritchken and Sankarasubramanian (1995) are summarised in the appendix.

Option prices are computed using Monte Carlo simulation under the risk-adjusted distribution described in the appendix. The speed of convergence is enhanced using control variates developed by obtaining analytical solutions for claims when $\gamma = 0$. In all our experiments the prices generated using the control variable stabilise to within 1% of the value by 2,000 iterations. The same accuracy without control variates requires 10,000 iterations.[6]

Comparison of option prices produced by different volatility structures

The models generated by different parameter values σ, κ and γ all share a common term structure. This permits us to investigate the sensitivity of option prices to changes in the parameter γ without altering the set of prices of the underlying bonds. Of course, in order to compare competing models of interest rate-sensitive claims, it is also important

Table 1. Comparison of prices of six-month call options on 15-year discount bonds

σ_0	κ	γ	Strike 0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.0	11.39	7.29	4.16	2.08	0.90
		0.5	11.41	7.31	4.16	2.06	0.88
		1.0	11.44	7.33	4.16	2.05	0.86
		1.5	11.46	7.35	4.17	2.03	0.84
	0.05	0.0	10.90	6.43	3.12	1.19	0.33
		0.5	10.91	6.45	3.12	1.18	0.33
		1.0	10.93	6.47	3.12	1.17	0.32
		1.5	10.95	6.48	3.12	1.15	0.31
0.010	0.01	0.0	14.49	11.14	8.32	6.03	4.25
		0.5	14.56	11.18	8.32	6.00	4.18
		1.0	14.63	11.22	8.32	5.97	4.12
		1.5	14.70	11.26	8.33	5.94	4.06
	0.05	0.0	12.81	9.17	6.24	4.02	2.44
		0.5	12.87	9.21	6.24	3.98	2.38
		1.0	12.93	9.25	6.25	3.95	2.33
		1.5	13.00	9.29	6.25	3.92	2.28
0.015	0.01	0.0	18.19	15.15	12.47	10.15	8.17
		0.5	18.30	15.20	12.47	10.09	8.06
		1.0	17.41	15.26	12.47	10.04	7.95
		1.5	18.52	15.32	12.47	9.99	7.85
	0.05	0.0	15.39	12.13	9.36	7.05	5.20
		0.5	15.50	12.19	9.36	7.00	5.10
		1.0	15.60	12.26	9.36	6.95	5.00
		1.5	15.71	12.32	9.37	6.91	4.91

X is the six-month forward price on the underlying bond. The notional principal for all options is set at US$1,000. The volatilities of 0.005, 0.01 and 0.015 refer to the instantaneous volatility parameter, σ, in the $\gamma = 0$ model. The first rows of numbers where $\gamma = 0$ refer to the benchmark option price.
For each γ ($\gamma \neq 0$), the parameter σ is selected so as to match the initial volatility of spot rates under the benchmark model. For example, when the volatility is 1%, the mean-reversion parameter, κ, is 0.05 and $\gamma = 0.5$, an option that is 2.5% in-the-money would be priced at US$9.21 and the benchmark price would be US$9.17.

that the *initial* volatilities of all bond prices be identical in the different models. With this accomplished, differences in prices result directly from the difference in assumptions regarding the *structure* of volatilities.

Let κ_0, σ_0 and γ_0 represent benchmark parameters and let $r(0)$ be the initial spot rate. Let κ, σ and γ be the parameters of an alternative model. Ensuring that all initial volatilities in the term structure are common across the different models requires that the parameters be restricted as

$$\sigma[r(0)]^{\gamma} e^{-\kappa T} = \sigma_0 [r(0)]^{\gamma_0} e^{-\kappa_0 T} \; \forall \geq 0 \tag{4}$$

This implies that $\kappa = \kappa_0$ and $\sigma[r(0)]^{\gamma} = \sigma_0[r(0)]^{\gamma_0}$.

The set of parameter values that we choose for γ are 0, 0.5, 1.0 and 1.5. As discussed earlier, $\gamma = 0$ corresponds to the generalised Vasicek model, $\gamma = 0.5$ corresponds to a square root model of volatility, similar to that of Cox, Ingersoll and Ross (1985), and $\gamma = 1$ is similar to the lognormal volatility considered by Dothan (1978). The value of $\gamma = 1.5$ is motivated by the parameter estimates obtained by Chan *et al* (1992) in their unrestricted models.

We chose the values of σ to ensure that the spot rate volatility, $\sigma[r(0)]^{\gamma}$, equals 0.5%, 1% or 1.5%. This range of values is reasonable and consistent with empirical experience.

The values of κ are taken to be 0.01 and 0.05. It is believed that in actuality this value is close to zero. Ball and Torous (1993) noted that when κ is near zero the interest rate process resembles a non-stationary process and that estimates will in general not be precise. Researchers have used a variety of techniques to estimate this parameter, often within the confines of the Cox, Ingersoll and Ross model, and confirm the fact that this parameter is difficult to estimate. Jegadeesh (1994) investigated a generalised Vasicek

Table 2. Comparison of percentage price differences of six-month call options on 15-year discount bonds relative to the $\gamma = 0$ benchmark

					Strike		
σ_0	κ	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.5	0.21	0.25	0.04	−0.72	−2.30
		1.0	0.43	0.50	0.08	−1.44	−4.57
		1.5	0.65	0.76	0.13	−2.15	−6.79
	0.05	0.5	0.15	0.26	0.05	−1.14	−4.00
		1.0	0.30	0.51	0.09	−2.26	−7.86
		1.5	0.46	0.77	0.14	−3.37	−11.60
0.010	0.01	0.5	0.48	0.37	0.04	−0.55	−1.51
		1.0	0.97	0.75	0.08	−1.08	−2.98
		1.5	1.45	1.12	0.13	−1.60	−4.42
	0.05	0.5	0.50	0.43	0.06	−0.80	−2.31
		1.0	1.00	0.87	0.12	−1.58	−4.57
		1.5	1.51	1.31	0.19	−2.35	−6.78
0.015	0.01	0.5	0.60	0.37	−0.01	−0.54	−1.33
		1.0	1.20	0.76	0.00	−1.06	−2.60
		1.5	1.81	1.15	0.01	−1.55	−3.85
	0.05	0.5	0.69	0.50	0.03	−0.72	−1.92
		1.0	1.38	1.01	0.08	−1.42	−3.79
		1.5	2.08	1.52	0.13	−2.09	−5.59

X is the six-month forward price on the underlying bond. The notional principal for all options is set at US$1,000. The volatilities of 0.005, 0.01 and 0.015 refer to the instantaneous volatility parameter, σ, in the $\gamma = 0$ model.
For each γ ($\gamma \neq 0$), the parameter σ is selected so as to match the initial volatility of spot rates under the benchmark model. For example, when the volatility is 1%, the mean-reversion parameter, κ, is 0.05 and $\gamma = 0.5$, an option that is 2.5% in-the-money would deviate from the benchmark model by 0.43%.

model and reported results for $\kappa = 0$, 0.05 and 0.1 that are all within his confidence interval of [−0.62, 0.40].[8]

Using daily estimates of Treasury yields for a 10-year period from December 1984 to February 1995, we estimate the volatilities of consecutive forward rates. The rate of decay in forward-rate volatilities is then computed. Our cursory findings suggest that, over all two-year sub-intervals of the data, the range in κ values is between 0.02 and 0.07, with almost all cases in the interval [0.02, 0.04].

For $\gamma = 0$, the variance of the spot rate one year into the future is obtained as

$$\sigma^2 \int_0^1 e^{-2\kappa(1-t)} dt = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa}\right]$$

For the range of values considered, the standard deviations of interest rates one year into the future are then found to be between 49 and 150 basis points. Further, with $\kappa = 0.05$, the volatilities of six-month, five-year and 30-year forward rates are 97.5%, 78% and 22% of the volatility of the current spot rate.

In all the numerical simulations the initial term structure is taken to be flat at 10%. The simulations exploit control variates, and 10,000 iterations were performed to obtain each price. In all cases this number is large enough to ensure that the standard errors of the estimated prices are sufficiently small to warrant ignoring them.

We first examine the sensitivity of short-term debt and interest rate options to alternative specifications of the elasticity parameter, γ. Tables 1 and 2 present six-month option prices on the 15-year discount bond and the percentage differences from the benchmark $\gamma = 0$ prices for a wide range of different parameter values of κ_0 and σ_0.

From Table 1 we note that *deviations* in the values of options from their benchmark values appear to expand with the elasticity parameter, γ, especially for away-from-the-money options. The direction of the deviation, however, tends to change according to whether the option is in- or out-of-the-money. From Table 2 we see that the differences in option prices generated by the different models may occasionally be large. The largest *absolute* differences arise when the volatility parameter is high and when the mean-reversion parameter is low.

For example, when $\sigma_0 = 0.015$, $\kappa_0 = 0.01$ and $\gamma = 1.5$, the difference in the price of the at-the-money call option from its $\gamma = 0$ counterpart is 3.85%. For this choice of parameter values, the six-month standard deviation is around 1.0%. For options that are in-the-money, however, the effect of the choice of γ is relatively small, with the exception of the extreme parameter cases. This suggests that for in- and at-the-money options with maturities of six months or so the error induced by an improper choice of the elasticity parameter is likely to be small.

For options that are deep out-of-the-money, variations in the tails of the different distributions can cause pricing deviations. This is seen in Table 2, where the pricing errors resulting from an improper choice of the elasticity parameter can be significant.

Tables 3 and 4 provide the same analysis for options on the spot interest rate, and similar conclusions can be drawn.

Markets for longer-term debt and interest rate options are becoming more active. Tables 5 and 6 show the sensitivity of five-year option prices on 15-year discount bonds to alternative elasticity values in the volatility structure, while Tables 7 and 8 repeat the analysis when the underlying security is the spot interest rate.

The *absolute* deviations from the inelastic model tend, for the most part, to be magnified because of the longer maturity. Further, unlike the results with short-term options, the pricing errors for in- and at-the-money options are also potentially severe. Our results suggest that when dealing with longer-term options, such as the call feature embedded in long-term callable bonds, greater emphasis should be placed on proper selection of the elasticity of spot rate volatilities.

The pricing of options on risky assets with stochastic interest rates

The pricing of options on risky assets has been extensively investigated, and the original Black–Scholes model has been extended to incorporate interest rate risk in one form or the other.[9] All existing state variable models that permit the term structure and volatilities to be initialised have the restriction that the volatility of interest rates is inelastic to the level of the term structure (ie, $\gamma = 0$). In what follows, we permit γ to deviate from zero and investigate the potential bias created when this elasticity parameter is ignored.

Our primary focus here is on longer-term stock options, where the effect of ignoring interest rate risk may be considerable. Indeed, such an analysis has important ramifications for pricing stock warrants, which are typically issued with maturities in the three- to 10-year range and are often priced ignoring interest rate risk altogether.

Assume the stock price dynamics are of the form

[Equation set in this form to pull back last line on this page from page 11.]

$$dS(t)/S(t) = \mu_S(\cdot)\,dt + \sigma_S\,dv(t)$$

and the interest rates evolve as described earlier. The instantaneous correlation between interest rates and stock returns is given by $E[dw(t)dv(t)] = \rho dt$. For simplicity, we assume that the underlying security corresponds to a financial asset that pays no dividends or coupons. Standard arbitrage arguments lead to the usual partial differential equations and the risk-neutral distributions for pricing all European claims. In particular, the risk-neutral distributions for interest rates remain unaltered, while the evolution of the asset price under the risk-neutral process is obtained by setting the drift term, $\mu_s(\cdot)$, to be equal to the instantaneous spot interest rate, $r(t)$.

Monte Carlo simulation techniques can then be used to price call options on the asset. As before, the analytical expressions that result when $\gamma = 0$ are used as control variates to reduce pricing errors.

Table 9 presents the results for six-month and five-year stock option contracts when interest rate risk is considered, when the correlation, ρ, between asset returns and interest rate movements is -0.5 and the annual volatility of the asset, σ_s, is 20%. The results indicate that for short-term options the effect of γ is not significant. This suggests that models of stock options that incorporate interest rate uncertainty by an inelastic, deterministic volatility structure, as in Amin and Jarrow (1992), are likely to provide acceptable results for short-term options on risk assets.

Table 3. Comparison of prices of six-month call options on the short interest rate

			Strike				
σ_0	κ	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.0	4.87	2.85	1.34	0.47	0.12
		0.5	4.87	2.84	1.34	0.48	0.12
		1.0	4.86	2.84	1.34	0.49	0.13
		1.5	4.86	2.83	1.34	0.49	0.14
	0.05	0.0	4.87	2.84	1.33	0.46	0.11
		0.5	4.86	2.83	1.33	0.47	0.12
		1.0	4.86	2.83	1.32	0.47	0.13
		1.5	4.85	2.82	1.32	0.48	0.13
0.010	0.01	0.0	5.70	4.03	2.68	1.65	0.94
		0.5	5.67	4.02	2.68	1.67	0.97
		1.0	5.65	4.00	2.67	1.68	1.00
		1.5	5.62	3.98	2.67	1.70	1.02
	0.05	0.0	5.68	4.01	2.65	1.63	0.92
		0.5	5.65	3.99	2.65	1.64	0.95
		1.0	5.63	3.97	2.65	1.66	0.97
		1.5	5.60	3.96	2.65	1.67	1.00
0.015	0.01	0.0	6.83	5.32	4.02	2.94	2.08
		0.5	6.78	5.29	4.01	2.96	2.12
		1.0	6.73	5.26	4.01	2.98	2.16
		1.5	6.69	5.23	4.00	3.00	2.20
	0.05	0.0	6.80	5.28	3.98	2.90	2.04
		0.5	6.75	5.25	3.97	2.92	2.08
		1.0	6.70	5.22	3.97	2.94	2.12
		1.5	6.65	5.19	3.97	2.96	2.17

X is the six-month forward price on the underlying bond. The notional principal for all options is set at US$1,000. The volatilities of 0.005, 0.01 and 0.015 refer to the instantaneous volatility parameter, σ, in the γ = 0 model. The first rows of numbers where γ = 0 refer to the benchmark option price.
For each γ (γ ≠ 0), the parameter σ is selected so as to match the initial volatility of spot rates under the benchmark model. For example, when the volatility is 1%, the mean-reversion parameter, κ, is 0.05 and γ = 0.5, an option that is 2.5% in-the-money would be priced at US$3.99 and the benchmark price would be US$4.01.

Table 4. Comparison of percentage price differences of six-month call options on the spot interest rate relative to the γ = 0 benchmark

			Strike				
σ_0	κ	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.5	−0.13	−0.24	−0.02	1.34	5.11
		1.0	−0.26	−0.48	−0.05	2.69	10.37
		1.5	−0.39	−0.72	−0.07	4.05	15.80
	0.05	0.5	−0.13	−0.24	−0.02	1.36	5.19
		1.0	−0.26	−0.48	−0.05	2.73	10.55
		1.5	−0.38	−0.72	−0.07	4.11	16.09
0.010	0.01	0.5	−0.49	−0.43	−0.06	0.88	2.65
		1.0	−0.97	−0.86	−0.11	1.76	5.34
		1.5	−1.44	−1.29	−0.16	2.64	8.07
	0.05	0.5	−0.49	−0.43	−0.05	0.88	2.69
		1.0	−0.97	−0.86	−0.11	1.77	5.41
		1.5	−1.44	−1.29	−0.16	2.66	8.18
0.015	0.01	0.5	−0.73	−0.53	−0.10	0.74	1.98
		1.0	−1.45	−1.05	−0.19	1.49	3.98
		1.5	−2.16	−1.56	−0.27	2.23	6.00
	0.05	0.5	−0.73	−0.53	−0.10	0.75	2.00
		1.0	−1.45	−1.05	−0.18	1.51	4.02
		1.5	−2.15	−1.57	−0.27	2.26	6.07

X is the six-month forward interest rate. The notional principal for all options is set at US$1,000. The volatilities of 0.005, 0.01 and 0.015 refer to the instantaneous volatility parameter, σ, in the γ = 0 model.
For each γ (γ ≠ 0), the parameter σ is selected so as to match the initial volatility of spot rates under the benchmark model. For example, when the volatility is 1%, the mean-reversion parameter, κ, is 0.05 and γ = 0.5, an option that is 2.5% in-the-money would deviate from the benchmark model by 0.43%.

Table 5. Comparison of prices of five-year call options on 15-year discount bonds

σ_0	κ	γ	Strike 0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.0	11.82	9.90	8.19	6.71	5.44
		0.5	11.93	9.97	8.23	6.72	5.42
		1.0	12.04	10.04	8.26	6.71	5.38
		1.5	12.14	10.10	8.29	6.71	5.35
	0.05	0.0	9.54	7.43	5.65	4.19	3.02
		0.5	9.63	7.49	5.67	4.18	2.99
		1.0	9.72	7.55	5.70	4.17	2.95
		1.5	9.81	7.61	5.72	4.16	2.92
0.010	0.01	0.0	19.54	17.89	16.34	14.91	13.58
		0.5	19.81	18.07	16.46	14.95	13.56
		1.0	20.01	18.21	16.53	14.95	13.49
		1.5	20.19	18.31	16.56	14.92	13.39
	0.05	0.0	14.71	12.91	11.28	9.81	8.49
		0.5	14.93	13.06	11.36	9.82	8.44
		1.0	15.13	13.19	11.42	9.81	8.37
		1.5	15.31	13.30	11.46	9.79	8.29
0.015	0.01	0.0	27.32	25.82	24.40	23.04	21.76
		0.5	27.77	26.16	24.62	23.15	21.76
		1.0	29.97	26.26	24.63	23.06	21.58
		1.5	28.07	26.26	24.53	22.88	21.31
	0.05	0.0	20.06	18.42	16.88	15.46	14.13
		0.5	20.42	18.67	17.03	15.50	14.07
		1.0	20.66	18.81	17.07	15.44	13.92
		1.5	20.85	18.90	17.06	15.34	13.73

X is the five-year forward price on the underlying bond. The notional principal for all options is set at US$1,000. The volatilities of 0.005, 0.01 and 0.015 refer to the instantaneous volatility parameter, σ, in the $\gamma = 0$ model. The first rows of numbers where $\gamma = 0$ refer to the benchmark option price.
For each γ ($\gamma \neq 0$), the parameter σ is selected so as to match the initial volatility of spot rates under the benchmark model. For example, when the volatility is 1%, the mean-reversion parameter, κ, is 0.05 and $\gamma = 0.5$, an option that is 2.5% in the money would be priced at US$13.06 and the benchmark price would be US$12.91.

Table 6. Comparison of percentage price differences of five-year call options on 15-year discount bonds relative to the $\gamma = 0$ benchmark

σ_0	κ	γ	Strike 0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.5	0.92	0.74	0.45	0.02	−0.54
		1.0	1.80	1.43	0.84	−0.02	−1.14
		1.5	2.65	2.09	1.18	−0.10	−1.80
	0.05	0.5	0.97	0.83	0.46	−0.19	−1.16
		1.0	1.91	1.63	0.87	−0.43	−2.35
		1.5	2.83	2.41	1.25	−0.69	−3.57
0.010	0.01	0.5	1.35	1.06	0.72	0.30	−0.19
		1.0	2.40	1.82	1.12	0.28	−0.71
		1.5	3.30	2.38	1.32	0.08	−1.39
	0.05	0.5	1.52	1.18	0.73	0.15	−0.60
		1.0	2.86	2.14	1.23	0.04	−1.46
		1.5	4.10	2.98	1.59	−0.21	−2.43
0.015	0.01	0.5	1.64	1.30	0.91	0.47	0.01
		1.0	2.40	1.71	0.94	0.07	−0.86
		1.5	2.74	1.71	0.56	−0.72	−2.09
	0.05	0.5	1.79	1.37	0.85	0.28	−0.44
		1.0	3.00	2.13	1.10	−0.08	−1.48
		1.5	3.95	2.60	1.04	−0.74	−2.79

X is the five-year forward price on the underlying bond. The notional principal for all options is set at US$1,000. The volatilities of 0.005, 0.01 and 0.015 refer to the instantaneous volatility parameter, σ, in the $\gamma = 0$ model.
For each γ ($\gamma \neq 0$), the parameter σ is selected so as to match the initial volatility of spot rates under the benchmark model. For example, when the volatility is 1%, the mean-reversion parameter, κ, is 0.05 and $\gamma = 0.5$, an option that is 2.5% in-the-money would deviate from the benchmark model by 1.18%.

Table 7. Comparison of prices of five-year call options on the short interest rate

					Strike		
σ_0	κ	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.0	4.43	3.47	2.64	1.95	1.40
		0.5	4.39	3.45	2.63	1.96	1.42
		1.0	4.36	3.42	2.63	1.97	1.45
		1.5	4.32	3.40	2.62	1.98	1.47
	0.05	0.0	4.22	3.23	2.40	1.72	1.18
		0.5	4.18	3.21	2.40	1.73	1.21
		1.0	4.15	3.19	2.39	1.74	1.23
		1.5	4.12	3.17	2.38	1.75	1.25
0.010	0.01	0.0	6.93	6.07	5.28	4.56	3.90
		0.5	6.84	6.01	5.25	4.56	3.93
		1.0	6.73	5.93	5.20	4.55	3.96
		1.5	6.61	5.84	5.14	4.52	3.97
	0.05	0.0	6.47	5.60	4.80	4.08	3.44
		0.5	6.38	5.54	4.77	4.08	3.47
		1.0	3.28	5.47	4.73	4.08	3.50
		1.5	6.17	5.39	4.68	4.06	3.52
0.015	0.01	0.0	9.53	8.70	7.92	7.18	6.49
		0.5	9.33	8.55	7.82	7.13	6.50
		1.0	9.10	8.37	7.69	7.05	6.47
		1.5	8.80	8.11	7.48	6.90	6.36
	0.05	0.0	8.82	7.98	7.20	6.47	5.79
		0.5	8.64	7.85	7.11	6.43	5.80
		1.0	8.44	7.69	7.00	6.37	5.79
		1.5	8.16	7.47	6.82	6.24	5.71

X is the five-year forward interest rate. The notional principal for all options is set at US$1,000. The volatilities of 0.005, 0.01 and 0.015 refer to the instantaneous volatility parameter, σ, in the $\gamma = 0$ model. The first rows of numbers where $\gamma = 0$ refer to the benchmark option price.
For each γ ($\gamma \neq 0$), the parameter σ is selected so as to match the initial volatility of spot rates under the benchmark model. For example, when the volatility is 1%, the mean-reversion parameter, κ, is 0.05 and $\gamma = 0.5$, an option that is 2.5% in-the-money would be priced at US$5.54 and the benchmark price would be US$5.60.

Table 8. Comparison of percentage price differences of five-year call options on the spot interest rate relative to the $\gamma = 0$ benchmark

					Strike		
σ_0	κ	γ	0.950X	0.975X	1.0000X	1.025X	1.050X
0.005	0.01	0.5	−0.77	−0.62	−0.21	0.57	1.82
		1.0	−1.55	−1.26	−0.46	1.08	3.60
		1.5	−2.37	−1.95	−0.78	1.52	5.30
	0.05	0.5	−0.76	−0.61	−0.20	0.67	2.06
		1.0	−1.52	−1.25	−0.44	1.30	4.06
		1.5	−2.32	−1.93	−0.75	1.88	5.97
0.010	0.01	0.5	−1.40	−1.09	−0.63	0.00	0.85
		1.0	−2.90	−2.32	−1.44	−0.16	1.48
		1.5	−4.67	−3.84	−2.57	−0.71	1.71
	0.05	0.5	−1.37	−1.07	−0.60	0.08	1.08
		1.0	−2.85	−2.27	−1.36	0.02	1.98
		1.5	−4.57	−3.74	−2.43	−0.38	2.45
0.015	0.01	0.5	−2.07	−1.73	−1.27	−0.71	0.03
		1.0	−4.44	−3.78	−2.91	−1.78	−0.42
		1.5	−7.65	−6.77	−5.54	−4.00	−2.07
	0.05	0.5	−2.03	−1.66	−1.19	−0.57	0.23
		1.0	−4.32	−3.64	−2.73	−1.50	0.09
		1.5	−7.41	−6.50	−5.21	−3.45	−1.23

X is the five-year forward interest rate. The notional principal for all options is set at US$1,000. The volatilities of 0.005, 0.01 and 0.015 refer to the instantaneous volatility parameter, σ, in the $\gamma = 0$ model.
For each γ ($\gamma \neq 0$), the parameter σ is selected so as to match the initial volatility of spot rates under the benchmark model. For example, when the volatility is 1%, the mean-reversion parameter, κ, is 0.05 and $\gamma = 0.5$, an option that is 2.5% in-the-money would deviate from the benchmark model by 1.07%.

Table 9. Comparison of percentage price differences of short- and long-term call options on a stock relative to the $\gamma = 0$ benchmark

σ_0	κ	γ	Strike 800	900	1,000	1,100	1,200
0.005	0.01	0.5	−0.01 (+0.00)	−0.02 (+0.00)	−0.02 (+0.00)	−0.02 (+0.00)	−0.02 (+0.00)
		1.0	−0.03 (+0.00)	−0.03 (+0.00)	−0.03 (+0.00)	−0.04 (+0.00)	−0.04 (+0.00)
		1.5	−0.04 (+0.00)	−0.04 (+0.00)	−0.05 (+0.00)	−0.06 (+0.00)	−0.06 (+0.01)
	0.05	0.5	−0.01 (+0.00)	−0.01 (+0.00)	−0.02 (+0.00)	−0.02 (+0.00)	−0.02 (+0.00)
		1.0	−0.02 (+0.00)	−0.03 (+0.00)	−0.03 (+0.00)	−0.04 (+0.00)	−0.04 (+0.00)
		1.5	−0.04 (+0.00)	−0.04 (+0.00)	−0.05 (+0.00)	−0.05 (+0.00)	−0.05 (+0.01)
0.010	0.01	0.5	−0.05 (+0.00)	−0.06 (+0.00)	−0.06 (+0.00)	−0.07 (+0.00)	−0.08 (+0.01)
		1.0	−0.10 (+0.00)	−0.11 (+0.00)	−0.12 (+0.00)	−0.13 (+0.00)	−0.15 (+0.02)
		1.5	−0.15 (+0.00)	−0.16 (+0.00)	−0.18 (+0.00)	−0.20 (+0.00)	−0.23 (+0.02)
	0.05	0.5	−0.05 (+0.00)	−0.05 (+0.00)	−0.06 (+0.00)	−0.07 (+0.00)	−0.07 (+0.01)
		1.0	−0.09 (+0.00)	−0.10 (+0.00)	−0.11 (+0.00)	−0.13 (+0.00)	−0.14 (+0.02)
		1.5	−0.14 (+0.00)	−0.15 (+0.00)	−0.17 (+0.00)	−0.19 (+0.00)	−0.21 (+0.02)
0.015	0.01	0.5	−0.12 (+0.00)	−0.13 (+0.00)	−0.14 (+0.00)	−0.14 (+0.00)	−0.17 (+0.02)
		1.0	−0.23 (+0.00)	−0.24 (+0.01)	−0.26 (+0.01)	−0.27 (+0.00)	−0.31 (+0.03)
		1.5	−0.36 (+0.01)	−0.38 (−0.01)	−0.40 (−0.01)	−0.43 (+0.00)	−0.48 (+0.05)
	0.05	0.5	−0.11 (+0.00)	−0.12 (+0.00)	−0.12 (+0.00)	−0.13 (+0.00)	−0.15 (+0.02)
		1.0	−0.21 (+0.00)	−0.22 (−0.01)	−0.24 (−0.01)	−0.26 (+0.00)	−0.30 (+0.03)
		1.5	−0.33 (+0.00)	−0.35 (−0.01)	−0.38 (−0.01)	−0.40 (+0.00)	−0.46 (+0.05)

The initial value of the stock, S(0), is set at US$1,000 and the annual volatility, σ_S, is set at 20%. The volatilities of 0.005, 0.01 and 0.015 refer to the instantaneous volatility parameter, σ, in the $\gamma = 0$ model.
For each γ ($\gamma \neq 0$), the parameter σ is selected so as to match the initial volatility of spot rates under the benchmark model. For example, when the volatility is 1%, the mean-reversion parameter, κ, is 0.05 and $\gamma = 0.5$, an option that is 10% in-the-money would deviate from the benchmark model by −0.05%.
The numbers in parentheses are the percentage biases generated using a $\gamma = 0$ model in lieu of the correct model for the interest rate process in valuing six-month call options on the stock. The numbers not in parentheses are the corresponding biases for five-year call options.

Table 10. Comparison of prices of five-year call options on a stock

			Strike				
σ_0	κ	γ	800	900	1,000	1,100	1,200
0.005	0.01	0.0	522.72	469.03	418.57	371.75	328.77
		0.5	522.65	468.96	418.50	371.67	328.71
		1.0	522.59	468.89	418.43	371.60	328.65
		1.5	522.52	468.83	418.37	371.53	328.58
	0.05	0.0	522.64	468.91	418.41	371.53	328.52
		0.5	522.58	468.85	418.34	371.47	328.46
		1.0	522.52	468.78	418.28	371.40	328.40
		1.5	522.46	468.72	418.21	371.33	328.34
0.010	0.01	0.0	524.07	471.06	421.37	375.31	333.06
		0.5	523.80	470.80	421.10	375.05	332.80
		1.0	523.55	470.55	420.86	374.80	332.56
		1.5	523.29	470.30	420.60	374.55	332.31
	0.05	0.0	523.88	470.78	420.98	374.82	332.48
		0.5	523.63	470.54	420.74	374.58	332.24
		1.0	523.40	470.31	420.51	374.34	332.02
		1.5	523.16	470.07	420.27	374.10	331.78
0.015	0.01	0.0	525.66	473.41	424.53	379.29	337.81
		0.5	525.04	472.82	423.97	378.74	337.25
		1.0	524.48	472.28	423.45	378.25	336.75
		1.5	523.79	471.64	422.83	377.67	336.19
	0.05	0.0	525.33	472.93	423.88	378.48	336.85
		0.5	524.76	472.38	423.36	377.97	336.33
		1.0	524.24	471.88	422.87	377.50	335.85
		1.5	523.60	471.28	422.29	376.95	335.31

The initial value of the stock, S(0), is set at US$1,000 and the annual volatility, σ_S, is set at 20%. The volatilities of 0.005, 0.01 and 0.015 refer to the instantaneous volatility parameter, σ, in the $\gamma = 0$ model. The first rows of numbers where $\gamma = 0$ refer to the benchmark option price.
For each γ ($\gamma \neq 0$), the parameter σ is selected so as to match the initial volatility of spot rates under the benchmark model. For example, when the volatility is 1%, the mean-reversion parameter, κ, is 0.05 and $\gamma = 0.5$, an option that is 10% in-the-money would be priced at US$470.54 and the benchmark price would be US$470.78.

Table 9 also shows that although long-term options on risky assets are more sensitive to the elasticity parameter than shorter-term options, the magnitude of the differences is still small.

Ignoring interest rate risk altogether could lead to serious pricing errors. Table 10 shows the sensitivity of option prices to the range of interest rate risk parameters, κ and σ. Over a five-year period the consequences of ignoring interest rate risk altogether can be seen to be significant.[10] For example, in Table 10, when $\sigma_0 = 0.015$, $\kappa = 0.01$ and $\gamma = 0$ the at-the-money option is valued at US$424.53. At the other extreme, when $\sigma_0 = 0.005$ and $\kappa = 0.05$ the value is US$418.41, which represents a 1.5% deviation.

In summary, while ignoring interest rate risk may lead to pricing errors, the choice of different spot rate volatility structures may not be that important.

Consequences of using a misspecified inelastic model

Since simple analytical equations are available for most European claims on interest rate-sensitive claims when the elasticity parameter, γ, is zero, it would be advantageous if they could be used as a proxy even if the correct volatility structure is not inelastic. Our evidence suggests that option prices on interest rates and bonds can be quite sensitive to the elasticity parameter when the other parameters of the model are known. In practice, however, the parameters are unknown and are usually estimated using observed data.

To address this issue, we investigate the bias incurred when an inelastic ($\gamma = 0$) model is used to establish theoretical prices when the true volatility structure is elastic ($\gamma \neq 0$) and the true parameters are unknown. Specifically, we first generate "true" option prices from an elastic volatility structure. Then, using an implied estimation approach, we estimate the volatility parameters (κ and σ) in the inelastic model using the least squares criterion.

Table 11. Implied estimates and errors with a misspecified model

Contract	Strike	$\gamma = 0.5$	$\gamma = 1.0$	$\gamma = 1.5$
Six-month option on 15-year bond	0.950X	12.87 (12.80)	15.13 (14.62)	13.00 (12.76)
	0.975X	9.21 (9.16)	13.19 (12.82)	9.29 (9.12)
	1.000X	6.24 (6.23)	11.42 (11.19)	6.25 (6.19)
	1.025X	3.98 (4.00)	9.81 (9.71)	3.92 (3.96)
	1.050X	2.38 (2.43)	8.37 (8.40)	2.27 (2.40)
Five-year option on 15-year bond	0.950X	14.93 (14.67)	12.93 (12.78)	15.31 (14.56)
	0.975X	13.06 (12.87)	9.25 (9.15)	13.30 (12.76)
	1.000X	11.36 (11.24)	6.25 (6.21)	11.46 (11.12)
	1.025X	9.82 (9.77)	3.95 (3.99)	9.79 (9.25)
	1.050X	8.44 (8.45)	2.33 (2.42)	8.28 (8.33)
Six-month option on short rate	0.950X	5.65 (5.69)	6.28 (6.49)	5.60 (5.70)
	0.975X	3.99 (4.02)	5.47 (5.62)	3.96 (4.03)
	1.000X	2.65 (2.66)	4.73 (4.82)	2.65 (2.68)
	1.025X	1.64 (1.64)	4.08 (4.10)	1.67 (1.65)
	1.050X	0.95 (0.93)	3.50 (3.45)	1.00 (0.95)
Five-year option on short rate	0.950X	6.38 (6.48)	5.63 (5.70)	6.17 (6.49)
	0.975X	5.54 (5.61)	3.97 (4.03)	5.39 (5.62)
	1.000X	4.77 (4.81)	2.65 (2.67)	4.68 (4.82)
	1.025X	4.08 (4.09)	1.66 (1.65)	4.06 (4.10)
	1.050X	3.47 (3.45)	0.97 (0.94)	3.52 (3.46)

The true prices generated by a $\gamma \neq 0$ model are reported in their respective columns. The estimated prices using a $\gamma = 0$ model are provided in parentheses. X is the forward price of the underlying instrument and the notional principal is set at US$1,000 in all cases. The true prices are simulated using 10,000 paths and control variate techniques. The resulting standard errors of the estimates are less than 0.001. The initial yield curve was flat at 10%, κ was chosen at 0.05 and $\sigma[r(0)]^{\gamma}$ was chosen at 1%.
For each $\gamma \neq 0$, the implied parameters using a $\gamma = 0$ model are chosen to minimise the sum of squared percentage errors. The resulting prices are in parentheses. For example, when $\gamma = 1.5$, the observed price of a six-month option on the short interest rate that is 2.5% in-the-money would be US$3.96. The estimated price using a $\gamma = 0$ model to minimise the sum of squared percentage errors across all options on bonds and rates would be US$4.03. This represents an error of 1.77%.

In particular, for each value of γ ($\neq 0$), using the 20 contracts shown in Table 11 we generate true option prices and use them to estimate the parameters of a $\gamma = 0$ model. The estimates are chosen to minimise the sum of squared percentage errors between the "true" ($\gamma \neq 0$) and "fitted" ($\gamma = 0$) prices. Once this is accomplished, the prices of the 20 contracts are computed using the simple analytical equations and compared to the true prices.

Table 11 shows the prices of contracts generated by three elastic models corresponding to γ values of 0.5, 1.0 and 1.5, together with their best fitted values resulting from a $\gamma = 0$ model. The results illustrate the degree of potential mispricing when a simple inelastic model is used to approximate option prices. As expected, the potential error tends to expand as γ increases. For example, the largest error of 16.4% arises when $\gamma = 1.5$ and a 2.5% in-the-money five-year option on a 15-year discount bond is considered.[11]

Since the volatility parameters are not known with certainty, estimation risk is present. If prices are very sensitive to these parameters, this source of uncertainty may dominate the error caused by misspecifying the elasticity parameter, γ. To investigate this issue, for each option contract in Table 11 we computed the adjustment to the volatility parameter, σ, from its current level that is needed to obtain the correct price. Table 12 displays the results.

As an example, consider the deep in-the-money six-month option on the 15-year bond and assume $\gamma = 1.5$. The volatility parameter has to be changed by 4.91% to price this contract correctly – ie, to 0.951 of its previously estimated value. If the numbers in each column are all small and in the same direction, this would suggest that the problem of correctly identifying γ is dominated by estimation risk. Large deviations in both directions, on the other hand, indicate that the elasticity parameter is important.

From the columns of the table we can see that as γ deviates from zero the range of necessary adjustments tends to expand. For $\gamma = 0.5$, the adjustments are very small, but for $\gamma \geq 1$ the adjustments begin to diverge.

Table 12. Errors with a misspecified model in terms of volatility estimates

Contract	Strike	γ = 0.5	γ = 1.0	γ = 1.5
Six-month option on 15-year bond	0.950X	1.50	3.14	4.91
	0.975X	0.85	1.83	2.94
	1.000X	0.22	0.57	1.05
	1.025X	−0.37	−0.61	−0.74
	1.050X	−0.94	−1.74	−2.42
Five-year option on 15-year bond	0.950X	2.45	4.84	7.20
	0.975X	1.73	3.37	5.00
	1.000X	1.08	2.09	3.07
	1.025X	0.48	−0.88	1.27
	1.050X	−0.11	−0.27	−0.42
Six-month option on short rate	0.950X	−1.77	−3.47	−5.01
	0.975X	−1.12	−2.17	−3.09
	1.000X	−0.48	−0.90	−1.18
	1.025X	0.15	0.35	0.70
	1.050X	0.77	1.61	2.59
Five-year option on short rate	0.950X	−2.14	−4.36	−6.79
	0.975X	−1.50	−3.06	−4.82
	1.000X	−0.84	−1.76	−2.86
	1.025X	−0.17	0.39	−0.77
	1.050X	0.55	1.05	1.36

These numbers represent the pricing errors between the true and estimated prices (see Table 11) in terms of the volatility input, σ. For example, for a five-year at-the-money option on the short rate with $\gamma = 1.5$, from Table 11 we obtain the true price as US\$6.17, while the pricing using a $\gamma = 0$ model fitted to the data is US\$6.49. The true price of US\$6.17 can be obtained using the $\gamma = 0$ model provided that the estimate of σ is lowered by 2.86% – ie, to 0.974 of its previously estimated value. This number is provided in this table.

This analysis highlights the importance of correctly estimating the elasticity parameter. Since the errors induced by a misspecified model may be significant, the benefit of using a simple inelastic model is marginal unless the true elasticity parameter is very close to zero. Furthermore, the benefit of simple analytical solutions is small, especially given the relative efficiency of the numerical procedures that can be developed for the two-state, single-factor Markovian models with elastic volatility structures.

Conclusion

This chapter investigates the sensitivity of option contracts to alternative volatility specifications on the spot interest rate. In a departure from previous studies, the initial conditions on bond prices and initial volatilities are controlled so that differences in prices can be attributed solely to differences in the *structure* of volatilities. To conduct such an analysis, we use the Ritchken–Sankarasubramanian family of interest rate models, which have a parsimonious set of parameters, permit the pricing of interest rate claims off a common initial term structure and allow a wide variety of volatility structures to be analysed.

Using the models developed here, we find that even when the initial conditions are held constant option prices on interest rates and on bonds can be quite sensitive to the elasticity parameter in the volatility structure of spot rates. This is in contrast to the results suggested by Hull and White (1990).

Further, we find that application of the generalised Vasicek model to price claims can lead to significant errors if the true volatility of the spot rate does depend on its level. Not surprisingly, the magnitude of the potential errors increases as the degree of elasticity in interest rates expands.

Our findings emphasise the need for further empirical research on volatility structures for forward rates. Specifically, they suggest that estimating the elasticity parameter, γ, in models that incorporate information from the existing term structure is an important consideration for pricing debt and interest rate options. It remains for future empirical work to identify the appropriate estimate of the elasticity parameter and to measure its stability over time.

Replace underlined with: "in terms of the initial term structure at date 0, the spot rate at date t and a second state variable, $\phi(t)$,..."?

Appendix

Ritchken and Sankarasubramanian (1995) showed that when the volatility structure of forward rates is as described in Equation (2), the level of the term structure at any future date, t, can be represented in terms of the initial term structure at date 0 in terms of the spot rate at date t and a second state variable, $\phi(t)$, that captures information pertaining to the path of interest rates over the interval [0, t].

In particular, for any date $T > t$,

$$f(t,T) = f(0,T) + e^{-\kappa(T-t)}[r(t) - f(0,t)] + e^{-\kappa(T-t)}\beta(t,T)\phi(t)$$

and

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-\beta(t,T)[r(t)-f(0,t)] - \frac{1}{2}[\beta(t,T)]^2\phi(t)}$$

where

$$\phi(t) = \int_0^t \sigma^2[r(u)]^{2\gamma} e^{-2\kappa(t-u)} du$$

and

$$\beta(t,T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)$$

In the above, forward rates, $f(t,T)$, and bond prices, $P(t,T)$, at date t are completely characterised by the state variables $r(t)$ and $\phi(t)$ in conjunction with a given *initial* term structure at date 0. $r(t)$ is the spot interest rate at date t, while $\phi(t)$ is the second state variable, which is a weighted sum of the spot rate volatilities realised over the interval [0, t].

The valuation of European-style contingent claims can be represented as an expectation of the terminal values under a modified forward risk-adjusted process. In particular, let $C(0,T)$ be the price at date 0 of a European claim that matures at date T. Using standard arbitrage arguments, it can be shown that the price of such a claim is given by

$$C(0,T) = P(0,T)\,E[C(T,T)]$$

where the expectation is taken under the equivalent martingale measure

$$dr(t) = \mu(r,\phi,t)dt + \sigma[r(t)]^{\gamma}dw(t)$$

$$d\phi(t) = \left(\sigma^2[r(t)]^{2\gamma} - 2\kappa\phi(t)\right)dt$$

where

$$\mu(r,\phi,t) = \kappa[f(0,t) - r(t)] + d/dt\, f(0,t) + \phi(t) + \beta(t,T)\sigma^2[r(t)]^{2\gamma}$$

For further details see Ritchken and Sankarasubramanian (1995).

1 *Of course, the parameters of the competing models are estimated using a common set of data. Nevertheless, the sets of initial bond prices and forward-rate volatilities generated by the different models using the estimated parameters are not identical.*

2 *The structure for forward-rate volatilities refers to the analytical representation of volatilities as a function of their maturities. Models with different structures could share a common initial set of values. For example, the volatilities of all forward rates could be set independently of their levels at 1% (as in Ho and Lee, 1986). In a second structure, the volatilities of forward rates could be proportional to their forward rates. These two models have different structures but their initial volatilities could be the same.*

3 *Given this volatility structure for forward rates and given an observed initial term structure, the drift term* $\mu_f(t, T)$ *is uniquely determined by the absence of dynamic arbitrage opportunities, as shown by Heath, Jarrow and Morton (1992).*

4 *In the Vasicek model, for example, the volatility of all longer-maturity forward rates is described as* $\sigma e^{-\kappa(T-t)}$ *for some constant* κ *and* σ*. This is obtained in our model by setting* $\gamma = 0$*.*

5 *Cursory empirical evidence reported by Heath* et al *(1992) suggests that the volatilities of forward rates may not decay but instead are humped. This feature can be incorporated by permitting* κ *to be a time-varying function. While this modification does not introduce any mathematical complications into our analysis, it is unlikely that any qualitative differences from our results will result from permitting* κ *to be a time-varying function.*

6 *For a discussion of control variate techniques and their use in pricing options see Boyle (1977) and Hull and White (1988). A list of the analytical solutions used as the control variates is available on request from the authors. Extensive simulations were performed to ensure that the discretisation process was fine enough to yield accurate prices.*

7 *For example, similar volatilities were reported by Jegadeesh (1994), who obtained an estimate of 1.1%, and by Barone, Cuoco and Zautzik (1991), who obtained an annualised volatility of* $\sigma_0 = 1.5\%$*.*

8 *For further empirical studies see Pearson and Sun (1992) and Gibbons and Ramaswamy (1993).*

9 *The value of European options on risky stocks in a stochastic interest rate economy has been considered for simple cases where interest rates are assumed to be lognormal or normal but not for the case where the volatility structure is of the general form in Equation (3). Examples of simple models include Merton (1973) and Rabinovitch (1989). Turnbull and Milne (1991) provide models of options on risky assets that permit the term structure to be initialised. Their models, however, require deterministic volatilities.*

10 *The importance of incorporating interest rate risk, evening short-term option models, should not be ignored. Indeed, the results for six-month options are, in percentage terms, quite similar.*

11 *Similar results are obtained over a large range of parameter values for the true* κ *and* σ *values.*

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