Risk and Return: The Portfolio Theory

The crux of portfolio theory - diversification:
- The risk (variance) on any individual investment can be broken down into two sources:
  - Firm specific risk (only faced by *that* firm),
  - Market wide risk (affects *all* investments).
- Firm-specific risk can be reduced, if not eliminated, by increasing the number of investments in your portfolio (i.e. by being diversified). Market wide risk cannot.
- On economic grounds, diversifying and holding a larger portfolio eliminates firm specific risk for two reasons:
  - Each investment is a much smaller percentage of the portfolio, muting the effect (positive or negative) on the overall portfolio.
  - Firm-specific actions can be either positive or negative. In a *large* portfolio, these effects will average out to zero.
- An individual holding a diversified portfolio should care about the *contribution* of a security to the *risk of the portfolio*, not the risk of that security on a *stand-alone* basis.
- The *contribution* of a security to the risk of a diversified portfolio should be rewarded by the market*, not the total risk of* the security (this contribution is measured by the "beta" of the security).

Portfolio Analytics

Example: Suppose two stocks A and B have the following returns distribution under 3 *equally likely* scenarios for the economy:

State of Economy	R_A	R_B
Boom	40%	10%
Normal	18%	-7%
Recession	-13%	24%

Then,

- The expected returns are:
    $$\overline{R}_A = 15\%, \qquad \overline{R}_B = 9\%$$
- The variance of returns is calculated as follows:

State	R_A	$R_A - \overline{R}_A$	$(R_A - \overline{R}_A)^2$	R_B	$R_B - \overline{R}_B$	$(R_B - \overline{R}_B)^2$
Boom	40%	25%	0.0625	10%	1%	0.0001
Normal	18%	3%	0.0009	-7%	-16%	0.0256
Recession	-13%	-28%	0.0784	24%	15%	0.0225
		Var(A)	0.0473		Var(B)	0.0161
		sd(A)	21.7%		sd(B)	12.7%

- Covariance: it's a measure of how two assets move together. A *positive covariance* suggests that their prices/returns move in the same direction most of the time, while a *negative covariance* suggests that they move in opposite directions.
- Correlation: a scaled measure of covariance between two random variables (scaled to 1). A correlation coefficient of 1 indicates perfect positive correlation, while -1 indicates perfect negative correlation.

- Computing covariance and correlation coefficient:

State	R_A	$R_A - \overline{R}_A$	R_B	$R_B - \overline{R}_B$	$(R_A - \overline{R}_A) \times (R_B - \overline{R}_B)$
Boom	40%	25%	10%	1%	0.0025
Normal	18%	3%	-7%	-16%	-0.0048
Recession	-13%	-28%	24%	15%	-0.0420
				Cov(R_A,R_B)	-0.0148
				ρ_{AB}	-0.54

- The covariance is given by
$$Cov(R_A, R_B) = \sigma_{AB} = E\left[(R_A - \overline{R}_A)(R_B - \overline{R}_B)\right]$$
- The correlation coefficient is given by
$$\rho_{AB} = \frac{Cov(R_A, R_B)}{\sigma_A . \sigma_B}$$
- These two stocks are negatively correlated, implying that the returns of A are higher in states where the returns of B are lower, and vice versa.
- Now, what would be the return and risk of a portfolio of 50%A and 50%B?

Portfolio Risk and Return

State	R_A	R_B	R_P	$R_P - \overline{R}_P$	$(R_P - \overline{R}_P)^2$
Boom	40%	10%	25%	13%	0.0169
Normal	18%	-7%	5.5%	-6.5%	0.004225
Recession	-13%	24%	5.5%	-6.5%	0.004225
Mean portfolio return			12%	Var(P)	0.00845
				sd(P)	9.2%

- The risk of the portfolio is lower than the risk of either of the two stocks!
- The expected return on a portfolio is given by the weighted average returns of the two assets
  $$\overline{R}_P = X_A \overline{R}_A + X_B \overline{R}_B$$
- The variance of returns of a portfolio is given by
  $$\sigma_P^2 = X_A^2 \sigma_A^2 + X_B^2 \sigma_B^2 + 2 X_A X_B \sigma_{AB}$$
- The covariance term affects the variance of the portfolio.
- Negative covariance between two securities decreases the variance of the entire portfolio.
- A kind of hedge - the two securities are offsetting each other.
- There is a diversification effect - the σ of the portfolio is not the weighted average σ of the two assets (in this example, the weighted average s.d. is 17.6%).
- There is no diversification effect if $\rho=1$, i.e., when the two assets are perfectly positively correlated.
- The same results apply for more than 2 assets.

The Efficient Portfolio

- What composition of A & B in the portfolio is "efficient"?
- What composition of A & B in the portfolio results in the minimum variance?
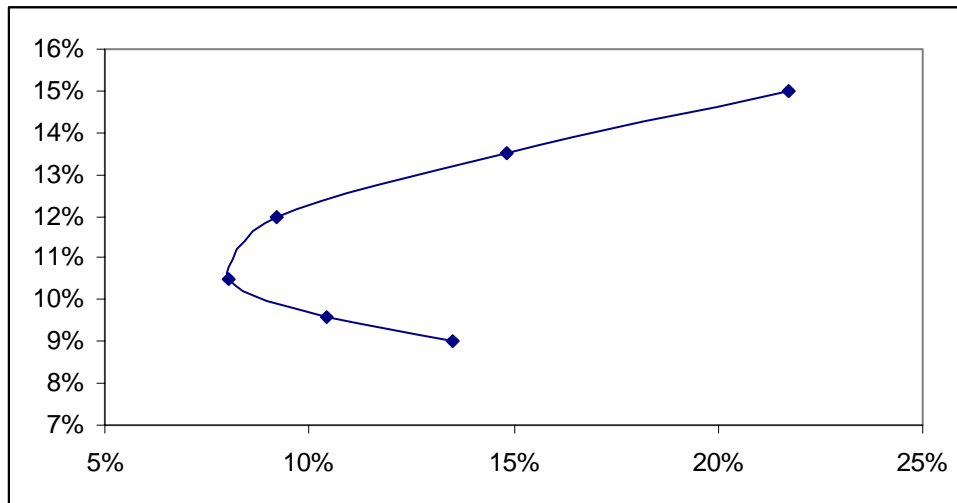
- The opportunity set, or feasible set consists of all possible risk return combinations for a portfolio of A & B.
- The efficient set, or the efficient frontier consists of risk-return combinations that rational investor would hold (the North-West frontier of the feasible set).
- The *Minimum Variance Portfolio (MVP)* reflects the *minimum* variance you can achieve using these two assets.
- The same results generalize for many securities.
- This framework is called the *mean-variance* framework, and these portfolios are called *Markowitz* portfolios.

Portfolio of Many Assets

- Suppose there are 'n' securities, with the same variance (var), and the covariance between any two securities is also the same (cov).
- Consider a portfolio in which all securities are equally weighted, i.e., the weight of each security is 1/n.
- From the variance covariance matrix, there are 'n' variance terms and n(n-1) covariance terms.
- The variance of the portfolio is given by:

$$Var(P) = \frac{1}{n}\text{var} + \left(1 - \frac{1}{n}\right)\text{cov}$$

- As 'n' goes to infinity, Var(P)→cov.
- This result has important implications - the variances of individual securities vanish as the number of securities becomes large.
- In other words, the variances of individual securities can be diversified away, but the covariance terms cannot be diversified.
- The portfolio variance can never drop to zero - it reduces to a floor of 'cov' (the average covariance across each pair of securities) as the number of securities increases.
- In practice, including about 30 stocks in a portfolio results in diversifying away about 99% of the diversifiable risk (there are transactions and other costs, so its not profitable to have portfolios with a very large number of assets!).

# The Break-up of Total Risk

- Total risk = portfolio risk + diversifiable risk
    (var)   (cov)   (var-cov)
- Total risk is the risk borne by holding one security only.
- Portfolio risk, or systematic risk), is the risk one still bears after achieving full diversification (cov).
- Diversifiable, or unsystematic risk, is the risk that can be diversified away in a large portfolio (var-cov).
- To an individual who has a diversified portfolio, the total risk of a security is not important - the contribution of the security to the risk of the portfolio is important, and that's what he will get rewarded for, not for the total risk.

Problems with the Mean-Variance Approach

- The Markowitz approach to portfolio optimization has two major problems:
  - It requires a very large number of inputs, since covariances between each pairs of assets are required to estimate the variance of the portfolio. E.g., if there are 2000 stocks, constructing an efficient portfolio would necessitate estimating about 4 million covariances!
  - It ignores a very important asset choice (in coming up with optimum portfolios) that most investors have - the choice of investing in a riskless security.
- The Capital Asset Pricing Model (CAPM) addresses these limitations and comes up with a far simpler approach to constructing optimum portfolios.

Riskless Borrowing and Lending

Example: Consider the Home Depot stock, with a 53.3% expected return and 68.5% volatility (estimated from historical data, 1990-94). If riskless lending and borrowing is available at 6.5%, then how can you combine the two to get portfolio returns and s.d.?
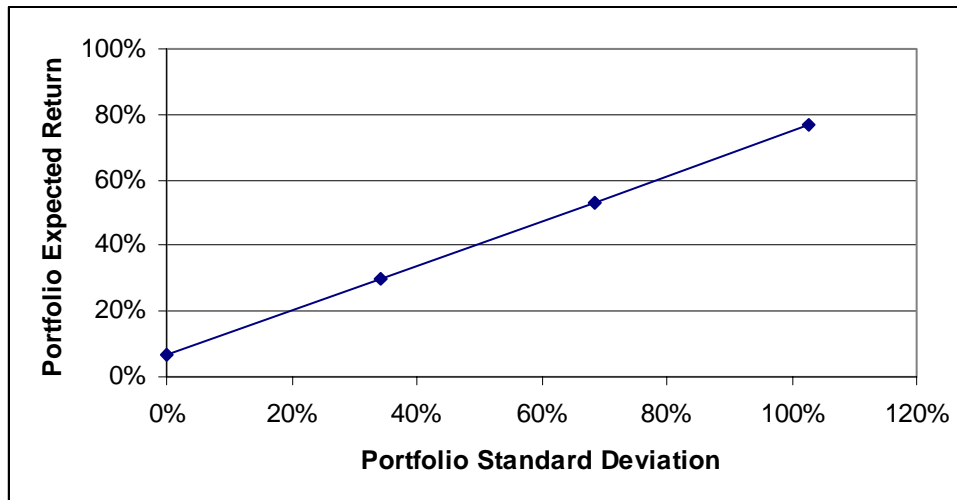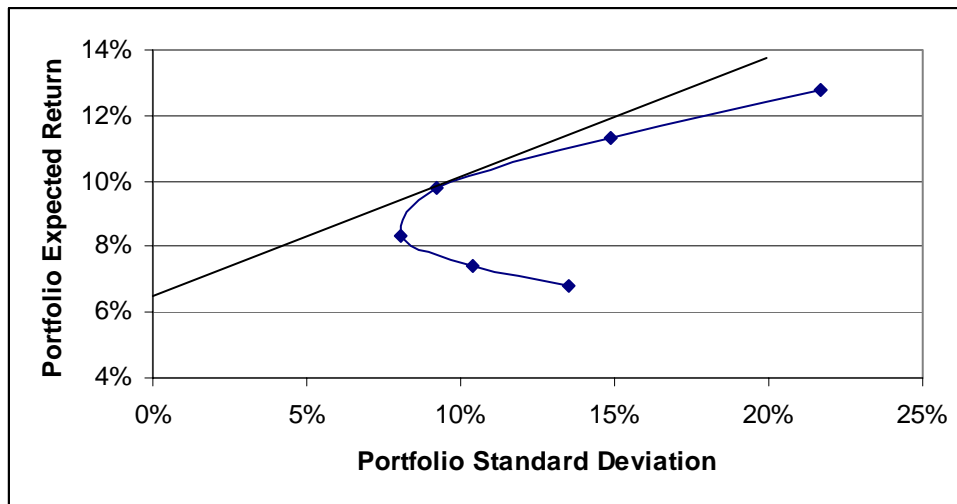

- Portfolio return and risk are given by
$$R_P = X_A R_A + (1 - X_A)R_F,$$
$$\sigma_P = X_A \sigma_A$$

- This is the equation of a straight line.
- What implications does it have for the efficient frontier, i.e., the set of efficient portfolios?

The Capital Market Line

- An investor can combine the riskless asset with a portfolio of any risky assets.
- What portfolio of risky assets would the investors use?

- The tangent line to the efficient set of risky assets provides investors with the highest return for any given standard deviation.
- If investors prefer more to less, and are risk-averse, they will prefer portfolios along the tangent line.
- The portfolio of risky assets that lies on the tangent line is the optimal risky portfolio, or the market portfolio (under the assumption of *homogeneous expectations*, where all investors will identify the same efficient set and the same capital market line).
- The tangent line is called the *Capital Market Line*.
- All investors will hold the *same* portfolio of risky assets, under these assumptions.
- In practice, a broad based index (like the S&P 500) is used as a proxy for the market portfolio.

The Implications

- Adding riskless assets to risky portfolios has strong implications for the optimal investment strategy .
- Implies that all investors will hold combinations of the riskless asset and the market portfolio.
- The only difference across investors is in the allocation decision - more risk-averse investors will invest more in the riskless asset, less risk-averse investors will invest more in the market portfolio.
- In this world, where investors hold a combination of only two assets - the riskless asset and the market portfolio, the risk of any asset will be measured relative to the market portfolio.
- Hence, the risk of any asset will be the risk it adds on to the market portfolio.
- The variance of the market portfolio prior to and after the addition of an individual asset is:

$$\text{var.\,prior to asset } i \text{ being added} = \sigma_{\mathrm{m}}^2$$

$$\text{var.\,after asset } i \text{ is added} = \omega_i^2 \sigma_i^2 + (1 - \omega_i)^2 \sigma_{\mathrm{m}}^2 + 2\omega_i(1 - \omega_i)\,\mathrm{cov}_{im}$$

$$\approx \sigma_{\mathrm{m}}^2 + 2\omega_i(1 - \omega_i)\,\mathrm{cov}_{im} \text{ (if } \omega_i \text{ is small)}$$

- The covariance term is the measure of risk added by asset *i,* i.e., the risk added by the new asset is proportional to its covariance with the market portfolio.

Beta (β)

- The beta of any investment is a standardized measure of the risk it adds to the market portfolio.

$$\beta_i = \frac{Cov(R_i, R_M)}{\sigma^2(R_M)}$$

- Beta is the measure of the risk of a security in a *large* portfolio (i.e., the non-diversifiable risk).
- Beta measures the responsiveness of a security to movements in the market portfolio.
- The beta of the market portfolio is one (why?).
- Assets that are riskier than average will have betas than exceed 1.
- Assets that are safer than average will have betas that are lower than 1.
- The beta for most stocks is between 0.7 and 1.3 - normally, financial stocks have high betas, and manufacturing stocks have low betas.
- A security with a beta of zero is called a *zero-beta asset* - it has no *relevant* risk (it's not the same as a risk-free security!).

Can any asset have a negative beta? what does it imply in terms of market risk and expected return?

The Capital Asset Pricing Model (CAPM)

- Beta is the appropriate measure of risk of a security in a large, diversified portfolio.
- Therefore, the expected return on a security should be positively related to its beta.
- CAPM posits a linear relationship between *expected* returns and security betas:

$$E(R_i) = R_f + \beta_i \left[E(R_M) - R_f \right]$$

- The plot of *E(R_i)* vs. β_i is called the *Security Market Line* (SML), which has an intercept equal to the risk free rate, and a slope equal to the market risk premium.
- Hence, higher *(systematic)* risk securities have higher expected returns.
- If the CAPM is correct, all securities should plot on the SML, which provides the expected return for any given beta (do they? what would happen if they don't?).

Example: What should be the expected return for a stock with a beta of 0.9 if the historical average return of the stock market were 12.5% and the Treasury bonds have an average yield of 6%?

Example: What would be the expected return on a portfolio consisting of 30% Exxon (β=0.65), 10% IBM (β=0.90), 15%AT&T (β=0.95), 15% GM (β=1.10), 5% GE (β=1.15), 20% Microsoft (β=1.30) and 5% Harley-Davidson (β=1.60), if the historical average return of the stock market were 12.5% and the Treasury bonds have an average yield of 6%?

The Real World: Limitations of CAPM

- Does the CAPM work?
- Is beta a good proxy for risk? (it should be correlated with expected returns if it is!).
- Roll (1977) showed that the CAPM *cannot be tested* as the market portfolio is unobservable (the market portfolio must consist of *all possible* investment alternatives that the investor has). Hence any test of the CAPM showed only if the model worked *given* the proxy (like the S&P 500 index) for the market.
- Fama and French (1992) showed that *firm size* and *book-to-market* ratio (ratio of book value of equity to market value of equity) explained differences in returns across firms much better than does beta, and may be better proxies for risk.
- Lead to the development of multi-factor model, where the measure of non-diversifiable risk comes from multiple factors, not just one factor as in the CAPM (the sensitivity to market portfolio).
- Still, the CAPM has been a widely used model on the Street, especially from the 60s till the early 90s.