Interest Rate Option Pricing With Volatility Humps*

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(January 1999)

*Peter Ritchken gratefully acknowledge financial support from the Federal Reserve Bank of Cleveland.

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Abstract

This paper develops a simple model for pricing interest rate options when the volatility structure of forward rates is humped. Analytical solutions are developed for European claims and efficient algorithms exist for pricing American options. The interest rate claims are priced in the Heath-Jarrow-Morton paradigm, and hence incorporate full information on the term structure. The structure of volatilities is captured without using time varying parameters. As a result, the volatility structure is stationary. It is not possible to have all the above properties hold in a Heath Jarrow Morton model with a single state variable. It is shown that the full dynamics of the term structure is captured by a three state Markovian system. Caplet data is used to establish that the volatility hump is an important feature to capture.

Keywords Interest Rate Claims, Volatility Humps.

Heath, Jarrow and Morton (1992) (hereafter HJM) have shown that the price of interest rate derivatives is fully determined by the volatility structure of forward rates. For rather general volatility structures, the dynamics of the term structure may be path dependent, a finite Markovian representation may not be possible, and pricing claims may become difficult. As a result, attention has focused on identifying restrictions on volatilities that lead to simple path independent and/or Markovian models of the term structure. For example, Caverhill (1994) identifies conditions that lead to finite state Markovian Gaussian models, and Ritchken and Sankarasubramanian (1995) provide conditions that lead to Markovian representations in which the volatility of all forward and spot rate depends on the level of the short term interest rate. When specific structures are assumed, simple algorithms for pricing claims can be developed. In the model building process, then, trade offs exist with more complexity in the volatility structure allowed at the cost of increased computational effort.

Empirical research suggests that the volatilities of forward rates may depend on their maturities. Several researchers report a hump in the volatility structure that peaks at around the two year maturity.[1] Introducing a volatility structure that is humped into the HJM paradigm is not that trivial. In particular, one of the simplest HJM models is the Gaussian Vasicek model, in which the volatilities decline exponentially with their maturity. Extending this model to capture a volatility hump can be accomplished at the expense of creating additional state variables. The question that naturally arises is whether the additional complexity of capturing this hump is warranted. This article addresses this issue.

The first goal of this article is to establish a simple model of the term structure that can capture a volatility hump. The second goal is to perform empirical tests on the model and to establish whether permitting a volatility hump is important. The model that we establish incorporates all current information in the yield curve and has the following properties. First, simple analytical solutions are available for most European claims. Second, the volatility of forward rates is humped, consistent with empirical evidence. Third, the volatility structure of forward rates is a stationary function, in that it only depends on the maturity of the rate.[2] Fourth, the model includes, as a special case, the generalized Vasicek models developed by Jamshidian (1989), HJM (1992) and Hull and White (1990), as well as the continuous time Ho-Lee (1988) model. Fifth, the model permits the efficient computation of American interest rate claims. Finally, the single factor models we present readily generalize to multifactor models.

The need for simple analytical solutions for European claims cannot be understated. In partic-

[1] A review of some empirical results is deferred to a later section.

[2] In particular, there are no time varying parameters in the model.

ular, an important property of any derivatives model is that it not only prices discount bonds at their observable values, but it also produces theoretical prices for an array of liquid derivatives that closely match their observable values. Typically, the calibration procedure is accomplished using the discount function as well as the prices of liquid caps and swaption contracts. In the HJM paradigm, all discount bonds will be automatically priced correctly. The parameters of the volatility structure, however, need to be determined so as to closely price a set of interest rate derivative contracts. This is usually accomplished by minimizing the sum of squared residuals. With many parameters, and with a highly non linear objective function, the optimization problem is non trivial, and multiple calls to valuation routines for the individual contracts arise. If these individual routines are not efficient, then implied estimation of the parameters becomes difficult. As a result, an important criterion for successful implementation is the ease in which the model's parameters can be readily calibrated. Since our simple model can easily be calibrated, it is likely to be more successful than a more complex model which might capture more precisely the volatility structure, but at the expense of forgoing analytical solutions and hence incurring costly calibrations.

The latter portion of the paper establishes the benefits of incorporating a volatility hump in a Generalized Vasicek model. Towards this goal, we perform empirical tests, using data on caplets that span the maturity spectrum, to establish whether incorporating a hump in the volatility structure is important.

The article proceeds as follows. In the next section we review the pricing mechanism in the HJM paradigm as well as the empirical evidence regarding the volatility hump. In section 2 we develop specific models for pricing European claims. We construct a two and three state-variable model, which includes as a special case the one state generalized Vasicek model. Analytical solutions for European options are provided. In section 3 we provide multivariate extensions to the model and we compare our one factor- three state variable model to a two factor-two state variable model developed by Hull and White (1994). In section 4 we develop efficient algorithms for pricing American claims. The algorithms are similar in spirit to those of Li, Ritchken and Sankarasubramanian (1995). Their model involves one source of uncertainty, yet requires two state variables. Here, we also have one source of uncertainty. However, up to three state variables are necessary to fully capture the dynamics of the term structure. By allowing these additional state variables to impact the yield curve provides a rich family of alternative models. We illustrate the convergence behavior of our algorithms. In section 5 we perform an empirical study on the model using caplet data. We find strong empirical support for the humped volatility structure of forward rates. In particular, we show that the Generalized Vasicek model is dominated by a model that permits a hump in the volatility structure. The stability of the parameter estimates over time is examined, and the maturity of the peak of the volatility is shown to be remarkably stable. Section 6 summarizes our findings and suggests directions for future research.

1 Pricing Mechanisms for Derivatives

Let $f(x,T)$ be the forward rate at date x for the instantaneous rate beginning at date T. Forward rates are assumed to follow a diffusion process of the form

$$df(x,T) = \mu_f(x,T)dt + \sigma_f(x,T)dw(x) \tag{1}$$

with the forward rate function $f(0,\bullet)$ initialized to its currently observable value. Here $\mu_f(x,T)$ and $\sigma_f(x,T)$ are the drift and volatility parameters which could depend on the level of the forward rate itself, and $dw(x)$ is the standard Wiener increment. HJM (1992) have shown that to avoid riskless arbitrage the drift term must be linked to the volatility term by:

$$\mu_f(x,T) = \sigma_f(x,T)[\lambda(x) + \sigma_p(x,T)] \tag{2}$$

where $\sigma_p(x,T) = \int_x^T \sigma_f(x,v)dv$ and $\lambda(x)$ is the market price of interest rate risk, which is independent of the maturity date T. Substituting equation (2) into (1) and integrating leads to

$$f(x,T) = f(0,T) + \int_0^x \sigma_f(v,T)[\sigma_p(v,T) + \lambda(v)]dv + \int_0^x \sigma_f(v,T)dw(v) \tag{3}$$

Now consider the pricing of an European claim that promises the holder a payout of $g(t)$ at date t. Here $g(t)$ is a cash flow fully determined by the entire term structure at that date. The arbitrage free price of this claim at date 0 is given by:

$$g(0) = E_0[g(t)]P(0,t) \tag{4}$$

where $P(0,t)$ is the price at date 0 of a bond that pays \$1 at date T. This expectation is computed under the *forward risk adjusted* process, which loosely speaking, is obtained by pretending $\lambda(v) = -\sigma_p(v,t)$ in equation (2). With this substitution, equation (3) can be written as:

$$f(t,T) = f(0,T) + h_1(t,T) + \int_0^t \sigma_p(v,T)dw(v) \tag{5}$$

where

$$\begin{aligned} h_1(t,T) &= \int_0^t g_1(v;t,T)dv \\ g_1(v;t,T) &= \sigma_f(v,T)\int_t^T \sigma_f(v,s)ds \end{aligned}$$

For pricing European claims it is usually easier to work under the forward risk adjusted process. In contrast, for pricing American claims, one usually proceeds by valuing under the *risk neutralized* process. In particular, as an alternative to equation (4), we have:

$$g(0) = \overline{E}_0[e^{-\int_0^t r(s)ds} g(t)] \tag{6}$$

The risk neutralized process can be viewed as a process where the market price of risk at date v is taken to be 0. Under this process, equation (3) reduces to

$$f(t,T) = f(0,T) + h_2(t,T) + \int_0^t \sigma_p(v,T)dw(v) \tag{7}$$

where

$$\begin{aligned} h_2(t,T) &= \int_0^t g_2(v;T)dv \\ g_2(v;T) &= \sigma_f(v,T)\sigma_p(v,T) \end{aligned}$$

From a valuation perspective, the HJM paradigm provides a framework where, given an initial term structure, the pricing mechanism can proceed once the volatility structure of forward rates is specified. The simplest volatility structure in the HJM paradigm is the constant volatility structure given as $\sigma_f(v,T) = \sigma$. This structure assumes all rates respond to a shock in the same way. Cursory empirical evidence suggests that volatilities of forward rates depend on their maturities. HJM (1993) and Jamshidian (1989) consider an exponentially dampened structure

$$\sigma_f(t,T) = \sigma e^{-\square(T-t)}.$$

This structure, referred to as the Generalized Vasicek or GV structure, implies that distant forward rates are much less volatile than near forward rates. If volatilities have this structure, then it can be shown that the entire dynamics of the term structure can be characterized by a *single* state variable, which could be the instantaneous spot rate, $r(t) = f(t,t)$, and that bond prices can be represented as

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-\beta(t,T)[r(t)-f(0,t)]-\frac{1}{2}\beta^2(t,T)\phi(t)} \tag{8}$$

where

$$\beta(t,T) = \frac{1}{\square}[1 - e^{-\square(T-t)}]$$

$$\phi(t) = \frac{\sigma^2}{2\square}[1 - e^{-2\square t}].$$

Ritchken and Sankarasubramanian (1995) show that if volatilities are not of this form then there is no single state variable HJM representation for the dynamics of the term structure.

There appears to be very little empirical support for an exponentially dampened forward rate volatility structure. Several researchers report a hump in the volatility structure that peaks at around the two year maturity. Litterman and Scheinkman (1991) use a principal component analysis of interest movements, to reveal a humped volatility form. Heath, Jarrow Morton and Spindel (1992) provide cursory evidence of such a hump. Amin and Morton (1994) use Eurodollar futures and options and obtain negative estimates of □ over the short end of the curve. Since negative estimates over the entire maturity spectrum are not plausible, they argue that there is a hump in the structure. Goncalves and Issler (1996) estimate the term structure of volatility using a simple GV model. Their historical analysis of forward rates also reveals a hump.[3] In addition to not providing for a volatility hump, GV models have the undesirable property that volatilities of yields are independent of their levels. As a result, interest rates can go negative. These problems have lead researchers to consider richer classes of volatility structures in which volatilities are linked directly or indirectly to the level of the term structure.

While the HJM paradigm permits the volatility structure to be quite general, unless constraints are imposed on the family of volatilities, a *finite* state representation of the term structure is not permissible. Ritchken and Sankarasubramanian (1995) characterize the set of restrictions on volatilities that permit a two state variable representation. In particular they show that if the volatility has the form

$$\sigma_f(t,T) = \sigma_r(t)k(t,T)$$

where $\sigma_r(t)$ is a function that depends on all information up to date t, and $k(t,T)$ is a deterministic function satisfying the following semi-group property:

$$\begin{aligned} k(t,T) &= k(t,u)k(u,T) \quad for\ t \quad u \quad T \\ k(u,u) &= 1 \end{aligned}$$

then, conditional on knowing the initial term structure, knowledge of any two points on the term structure at date t is sufficient to characterize the full yield curve at that date. The class of volatility structures in this family is quite large. However, no analytical solutions have been derived for European claims. As a result, calibration issues remain, which inhibit the easy implementation of these models.

Clearly there are disadvantages in maintaining a simple exponentially dampened Vasicek volatility structure; however, there are also significant difficulties in obtaining simple solutions for interest

[3]Not all studies indicate the existence of a hump. For example, Bliss and Ritchken (1995) use term structure data alone and find that relative to the volatility at the short end, forward rate volatilities appear to decline with maturity.

rate claims when forward rate volatilities are allowed to depend on forward rate levels. A compromise is to investigate Generalized Vasicek models where the number of state variables exceed the number of sources of uncertainty. The volatility structure of these types of models can be made to take on shapes other than the exponentially dampened structure. The model that we examine next belongs to this generalized family of models. In particular, we propose a model that allows volatilities of forward rates to have a humped term structure.

2 Option Pricing with a Volatility Hump

Assume the volatility structure is given by:

$$\sigma_f(x,T) = [a_0 + a_1(T-x)]e^{-\square(T-x)} + b_0 \tag{9}$$

This volatility function reduces to the GV structure when $a_1 = b_0 = 0$. Bhar and Chiarella (1995) have considered similar structures for volatilities. Indeed, they show that a *finite* state Markov representation is permissible for the term structure if the coefficient of the exponentially dampened term is a finite degree polynomial in the maturity $T-x$. Figure 1 shows a typical curve, in which the peak occurs around the two year point.

[Figure 1 Here]

For $0 \quad x \quad t$, the volatility structure can be expressed as:

$$\sigma_f(x,T) = d_0(t,T) + d_1(t,T)e^{-\square(t-x)} + d_2(t,T)[t-x]e^{-\square(t-x)} \tag{10}$$

where

$$\begin{aligned} d_0(t,T) &= b_0 \\ d_1(t,T) &= [a_0 + a_1(T-t)]e^{-\square(T-t)} \\ d_2(t,T) &= a_1 e^{-\square(T-t)} \end{aligned}$$

Substituting equation (10) into equation (5) yields:

$$f(t,T) = f(0,T) + h_1(t,T) + \sum_{i=0}^{2} d_i(t,T)W_i(t) \tag{11}$$

where

$$W_0(t) = \int_0^t dw(v) \tag{12}$$

$$
\begin{aligned}
W_1(t) &= \int_0^t e^{-\square(t-v)} dw(v) && (13) \\
W_2(t) &= \int_0^t (t-v) e^{-\square(t-v)} dw(v) && (14)
\end{aligned}
$$

and the exact expression for $h_1(t,T)$ is provided in the appendix.

Proposition 1 *If the volatility structure is given by equation (9), and the dynamics of the forward rates are given by equation(1), then, under the FRA process, bond prices at date t are linked to prices at date 0 through three state variables, $W_0(t)$, $W_1(t)$ and $W_2(t)$ as:*

$$
P(t,T) = A(t,T)e^{-R(t,T)} \tag{15}
$$

where

$$
A(t,T) = \frac{P(0,T)}{P(0,t)} e^{-H_1(t,T)}
$$

$$
R(t,T) = \sum_{i=0}^{2} D_i(t,T) W_i(t)
$$

and

$$
\begin{aligned}
H_1(t,T) &= \int_t^T h_1(t,x)dx \\
D_0(t,T) &= b_0(T-t) \\
D_1(t,T) &= \frac{1}{\square^2}[a_1 + a_0\square - (a_1\square(T-t) + a_0\square + a_1)e^{-\square(T-t)}] \\
D_2(t,T) &= \frac{a_1}{\square}[1 - e^{-\square(T-t)}]
\end{aligned}
$$

The dynamics of the state variables, $W_1(t)$, and $W_2(t)$ are:

$$
\begin{aligned}
dW_1(t) &= -\square W_1(t)dt + dw(t) && (16) \\
dW_2(t) &= [W_1(t) - \square W_2(t)]dt && (17)
\end{aligned}
$$

Proof: See Appendix.

When $b_0 = 0$, $D_0(t,T) = 0$, and the number of state variables reduces to 2. Further, when $a_1 = b_0 = 0$, then, the number of state variables reduces to 1, the GV volatility structure is recovered and the bond pricing equation reduces to equation (8).

Under the FRA process, viewed from date 0, the bond price, $P(t,T)$, has a lognormal distribution. In particular, $R(t,T)$ is normal with mean 0 and variance $\gamma^2(t,T)$, where:

$$
\gamma^2(t,T) = \sum_{j=0}^{2}\sum_{i=0}^{2} D_i(t,T)D_j(t,T)Cov_0(W_i(t), W_j(t)) \tag{18}
$$

and

$$
\begin{aligned}
Var_0(W_0(t)) &= t \\
Var_1(W_1(t)) &= \frac{1}{2\square}[1-e^{-2\square t}] \\
Var_2(W_2(t)) &= \frac{1}{4\square^3}[1-(1+2\square t+2\square^2 t^2)e^{-2\square t}] \\
Cov_0(W_0(t),W_1(t)) &= \frac{1}{\square}[1-e^{-\square t}] \\
Cov_0(W_0(t),W_2(t)) &= \frac{1}{\square^2}[1-(1+\square t)e^{-\square t}] \\
Cov_0(W_1(t),W_2(t)) &= \frac{1}{4\square^2}[1-(1+2\square t)e^{-2\square t}]
\end{aligned}
$$

We now can compute analytical solutions for a large family of European interest rate claims. Proposition 2 provides the solution to an European option on a discount bond.

Proposition 2 *If the volatility structure is given by equation (9), then the price of a contract that provides the holder with the right to buy at date t, a bond that matures at date T, for \$$X$ is given by:*

$$C(0) = P(0,T)N(d_1) - XP(0,t)N(d_2) \tag{19}$$

where

$$
\begin{aligned}
d_1 &= \frac{log(A(t,T)/X)+\gamma^2(t,T)}{\gamma(t,T)} \\
d_2 &= d_1 - \gamma(t,T)
\end{aligned}
$$

and $\gamma^2(t,T)$ is given by equation (18).

Proof: See Appendix.

Notice that when $a_1 = b_0 = 0$, the formula reduces to the GV option model of Jamshidian (1989).

It is possible to establish a simple interest rate claim model with one state variable that allows for a humped volatility structure. Moraledo and Vorst (1997), for example, show how such a model can be set up when the parameters are time varying. In particular, their model is obtained with $a_1 = b_0 = 0$ and $\square = \square(t) = \lambda - \gamma/(1+\gamma t)$. This model permits the hump, but has the property that the volatility structure is no longer stationary. That is the volatility does not depend on the maturity of the forward rate alone. If the volatility structure of forward rates is to be a humped deterministic function of the maturity of rates, then for an arbitrary initial term structure, it must be the case that the underlying number of state variables exceeds one. This fact follows from Caverhill (1994)and Ritchken and Sankarasubramanian (1995), among others.

3 Multifactor Models

The above analysis readily generalizes to multifactor models. In particular, we could consider models of the form

$$df(t,T) = \mu_f(t,T)dt + \sigma_{f_1}(t,T)dw_1(t) + \sigma_{f_2}(t,T)dw_2(t) \tag{20}$$

where the volatility structures, $\sigma_{f_1}()$ and $\sigma_{f_2}()$, have forms as in equation (9).

Analytical solutions for caplets under this process can be easily derived.

As an example, consider a specific two factor model in this family where:

$$\begin{aligned} \sigma_{f_1}(t,T) &= a_0 e^{-\square(T-t)} \\ \sigma_{f_2}(t,T) &= b_0 \\ E[dw_1 dw_2] &= \rho dt \end{aligned}$$

This model differs from simple two factor GV model, in that the two factors are correlated. Transforming this model, we obtain:

$$df(t,T) = \mu_f(t,T)dt + (a_0 e^{-\square(T-t)} + \rho b_0)d\xi_1(t) + (\sqrt{1-\rho^2})b_0 d\xi_2(t) \tag{21}$$

where $d\xi_1$ and $d\xi_2$ are standard independent Wiener increments with $E[d\xi_1 d\xi_2] = 0$. Moreover, under the forward risk adjusted process,

$$\mu_f(x,T) = \sum_{i=1}^{2} \int_0^t [\sigma_{f_i}(x,T) \int_x^T \sigma_{f_i}(x,s)ds]$$

Now, from equation (21)

$$f(t,T) = f(0,T) + h_1(t,T) + d(t,T)Z(t) + \rho b_0 \xi_1(t) + \sqrt{1-\rho^2} b_0 \xi_2(t)$$

where

$$\begin{aligned} d(t,T) &= a_0 e^{-\square(x-t)} \\ Z(t) &= \int_t^T e^{-\square(t-x)} d\xi_1(x) \\ h_1(t,T) &= \int_0^t \mu_f(x,T)dx \end{aligned}$$

Further, the bond price can be computed as

$$P(t,T) = A(t,T)e^{-R(t,T)}$$

where

$$
\begin{aligned}
A(t,T) &= \frac{P(0,T)}{P(0,t)} e^{-H_1(t,T)} \\
R(t,T) &= D(t,T)Z(t) + \rho b_0(T-t)\xi_1(t) + (\sqrt{1-\rho^2})b_0(T-t)\xi_2(t) \\
H_1(t,T) &= \int_t^T h_1(t,x)dx \\
D(t,T) &= \int_t^T d(t,x)dx
\end{aligned}
$$

This two factor model is characterized by three state variables. Similarly to Proposition 2, we can obtain the price of a contract that provides the holder with the right to buy at date t, a bond that matures at date T, for $\$X$ as:

$$
C(0) = P(0,T)N(d_1) - XP(0,t)N(d_2) \tag{22}
$$

where

$$
\begin{aligned}
d_1 &= \frac{log(A(t,T)/X) + \gamma^2(t,T)}{\gamma(t,T)} \\
d_2 &= d_1 - \gamma(t,T) \\
\gamma^2(t,T) &= Var(R(t,T))
\end{aligned}
$$

Analytical solutions for options on discount bonds can be obtained for the case where both volatility structures are of the form in equation (9). The general equation has the same form as the above option equation, except the variance expression in d_1 and d_2 are different.

Interestingly, by comparing the volatility expressions that differentiate a one factor model with a two factor model, it can be shown that prices of caplets from our two factor model will be very close to prices from our one factor model. Intuitively, this makes sense, since each caplet has a single cash flow, and if volatility for the one factor can explain uncertainty well, then the additional contribution by the second factor will be small. The observation that improvements in caplet pricing, when moving from a one factor to a two factor model, might be negligible has also been made by Rebonato (1996).

An alternative approach for pricing interest rate claims and ensuring a volatility hump, is to begin with a two factor, two state variable model, where, under the risk neutralized measure, we have:

$$
\begin{aligned}
dr &= [\theta(t) + u - ar]dt + \sigma_1 dw_1 \\
du &= \delta[\bar{\theta} - bu]dt + \sigma_2 dw_2
\end{aligned}
$$

where $E[dw_1 dw_2] = 0$. In this model, the interest rate follows a mean reverting process, whose central tendency is itself mean reverting. The economic rationale, for such models has been discussed by Jegadeesh and Pennacchi (1996), Balduzzi, Bertola and Foresi (1993), Hull and White (1994) and others. In such models, the logarithm of bond prices is affine in the state variables. Further, if the mean reversion and volatility parameters are curtailed the volatility structure for forward rates can be shown to be stationary and a humped function of maturity. Specifically, for the case where $\bar{\theta} = 0$, the volatility structure is:

$$\sigma_f(t,T) = \sigma_1 e^{-a(T-t)} + \frac{\sigma_2}{a-b}[e^{-b(T-t)} - e^{-a(T-t)}]$$

The forward rate volatility structure, while not identical to our structure, leads to an analytical solution for option prices on bonds that has the same form as our equation, but with a different variance expression.

In summary, for the pricing of caplets, our one factor model with up to three state variables leads to a similar form as a two stochastic driver model with two state variables, as in Hull and White. The advantages, or disadvantages of beginning with a dynamic of state variables under a risk neutralized measure, or beginning with a predefined volatility structure for forward rates, depends largely on the application. Jegadeesh and Pennacchi (1996), for example, set up a general equilibrium model, where the risk neutralized process is of the form in the above equations. Their model allows a time series analysis on Eurodollar futures prices to be conducted with the objective of extracting parameter estimates. Of special importance, was the estimation of the central tendency parameters, which typically display very large standard errors. In contrast, if the orientation is towards option pricing, then, in the Heath-Jarrow-Morton paradigm, the volatility structure is all important. Since caplet prices will be especially sensitive to the imposed volatility structure, as a function of maturity, and less sensitive to the number of stochastic drivers, using caplet information to understand volatility structures of forward rates is particularly useful. Indeed, in our empirical section we will use caplet prices to assess the importance of capturing the volatility hump. This is particularly important since the empirical evidence on the volatility hump is unclear.[4]

Of course, if the goal is to price swaptions, or other contracts that are especially sensitive to correlations among forward rates, then a model with two or more stochastic drivers might be useful. In this case, not only is it important to obtain a useful characterization of the volatility structure

[4]As discussed earlier, Amin and Morton find volatility of forward rates to be an increasing function of maturity, but their analysis is restricted to short term contracts. Goncalves and Issler (1996) find support for a decreasing volatility, over longer maturities. Taken together these two studies indicate a hump is present. Bliss and Ritchken (1996), however, provide some evidence that indicates the hump is not that important.

of all forward rates, but it also is essential to capture the correlation structure among forward rates. Given the dynamics of the two mean reverting processes, as above, the volatility structure of forward rates and their correlation structures come as given outputs. In contrast, in our two stochastic driver HJM model, as in equation (20), having two volatility structures that contain additional parameters, permit not only a stationary volatility hump to be fitted, but also provides more flexibility in attaining alternative stationary correlation structures.

4 Pricing American Options Under Humped Volatilities

The advantage of a simple deterministic volatility structure as in equation (9), is that it permits the development of analytical solutions for many European claims. The resulting expressions have the same form as their simple GV counterparts, except the volatility expression ($\gamma(t,T)$) takes on a more complex form. Analytical solutions for such claims are useful since they reduce the complexity of the calibration process. Once the parameters are estimated, then lattice based algorithms can be used to price a variety of American claims and some interest rate exotics. In this section we describe such an algorithm.

The valuation procedure takes place using the risk neutralized measure. Under this measure the term structure at date t is given by:

$$f(t,T) = f(0,T) + h_2(t,T) + \int_0^t \sigma_p(v,T)dw(v) \tag{23}$$

and the bond pricing equation is:

$$P(t,T) = A(t,T)e^{-R(t,T)} \tag{24}$$

where

$$\begin{aligned} A(t,T) &= \frac{P(0,T)}{P(0,t)} e^{-H_2(t,T)} \\ R(t,T) &= \sum_{i=0}^{2} D_i(t,T)W_i(t) \\ H_2(t,T) &= \int_t^T h_2(t,x)dx. \end{aligned}$$

Assume the time interval $[0,t]$ is partitioned into n equal subintervals of width h. A simple binomial lattice is used to approximate the standard Wiener process. Let $W^a_{0,i}$ approximate the process at time ih for $i = 0, 1, 2, ..., n$, with $W^a_{0,0} = 0$. Given $W^a_{0,i} = w$, the next permissible values are $w + \sqrt{h}$

and $w - \sqrt{h}$, which both occur with probability 0.5. For pricing purposes, the term structure can be recovered at each node of the lattice if the exact values of the three state variables are given. Let (W_1^a, W_2^a) be the values of the two state variables at a particular node, where the first state variable has value W_0^a. The number of different values for the state variables W_1^a and W_2^a at this node equals the number of different paths that can be traversed from the originating node to this point. Rather than keep track of all these values, we follow the basic idea of Li, Ritchken and Sakarasubramanian (1996) and only keep track of the maximum and minimum values that each of the two state variables can attain at each node. The range between the maximum and minimum values is then partitioned into k_1 and k_2 pieces respectively. We track option prices at the resulting $k_1 \times k_2$ points. Thus at each node in the lattice, a matrix of option values needs to be established. As the space partition values k_1 and k_2 increase, and as the number of time increments increase, the option prices converge to their true theoretical values.

Let $C(i, j)$ be the $(i, j)^{th}$ entry for a price that is to be computed at the node $W_0^a = w$ and assume $W_1^a = y$ and $W_2^a = z$. Assume the date is mh say, for some integer $m \quad (n-1)$. Given these two state variables, their successor values at date $(m+1)h$ can be computed using approximations to equations (16) and (17). In particular, the two successor nodes are $(w + \sqrt{h})$ and $(w - \sqrt{h})$ both of which occur with probability 0.5. The values of the two state variables, obtained using equations (16) and (17), at these two nodes, are $(y - (\square yh) + \sqrt{h}, z + (y - \square z)h)$ and $(y - (\square yh) - \sqrt{h}, z + (y - \square z)h)$ respectively. If option prices at all successor nodes, for the particular state variables, were available, then the first two moments of the true underlying dynamics over the time increment would be perfectly matched. In this case, convergence in distribution of the discrete time process to the continuous process is guaranteed. However, option prices at the successor nodes, for the particular updated state variables, may not be available. By construction, option prices at "surrounding" states will be available, and interpolation procedures can be used to establish an option price. The "average" of the option prices computed in both the up state and down state can then be computed, and the resulting value discounted by the current one period bond price, provides the value of the option unexercised at the current location. The maximum of this value and the exercised value of the claim provides the numerical value for $C(i, j)$. [5]

When computing option prices using backward recursion, various interpolation techniques can

[5] Since the use of an interpolation scheme for the additional state variables can be viewed as giving the increments a positive variance, the moments of the discretized distribution may not match up exactly with the true distribution. However, as the number of points between the maximum and minimums at each node increase, that is, as k_1 and k_2 increase, the moments will converge to their true values. Hence convergence in distribution is guaranteed as k_1, k_2, and $n \to \infty$.

be used to establish the values of the claim in both successor states. Li, Ritchken and Sankarasubramanian (1995), show that relatively coarse partitions of the range of the state variable at each node, combined with simple linear interpolation methods produce satisfactory results for their problem.[6]

We first report on the performance of an algorithm for the volatility structure in equation (9) with $a_1 = 0$. In this case there are only two state variables. Since analytical solutions are available for European options on bonds, we use these contracts to illustrate the convergence behavior for the contract as the number of time partitions, n, and as the number of space partitions, k_1, increase. Figure 2 reports the results for the one year at-the-money option, when a simple linear interpolation method is used.

[Figure 2 Here]

Notice, that for all time partitions, as k_1 increases the convergence rate improves. Notice too, that reasonably accurate results for option prices are obtained for 50 time partitions and about 5 space partitions.[7] For small values of k_1, and a large number of time periods the option prices might diverge. The reason for this is that linear interpolation between option prices at each node in the lattice may consistently overestimate the actual value, when the claim is convex in the state variable. Increasing the number of time periods may do little to improve the convergence properties. Convergence, however, is guaranteed to occur when the number of time periods, n increases, and when the space partition for the additional state variable is refined. Figure 2 clearly shows this pattern. Namely, when n is large convergence is not guaranteed unless k_1 is also large. As k_1 and n increase the prices on the lattice converge to the true theoretical price.

The convergence of option prices improves as k_1 increases. For a fixed value of k_1, the option prices appear to diverge as the number of time increments increases confirming the problems with linear interpolation. Since our contract has payouts that are convex in this state variable, quadratic interpolation should improve convergence. Table 1 compares the convergence rate of prices for various space partitions using a linear interpolation scheme, to the case where only 3 points are used to approximate the state variable at each node, but a quadratic approximation is invoked. As can be seen, the quadratic approximation works very effectively, producing accurate results even for a small number of time partitions.

[Table 1 Here]

[6]For example, they show that partitions of size 10 to 20 produce prices of options on bonds that are practically indistinguishable.

[7]Similar results hold over the full range of parameters for the volatility structure.

Figure 3 shows the convergence of option prices to the analytical solution for the general volatility structure with $a_1 \neq 0$. In Figure 3 the partition sizes (k_1 and k_2) for the two state variables $W_1(t)$ and $W_2(t)$ are taken to be equal.

[Figure 3 Here]

Table 2 compares the effects of using a quadratic approximation scheme with linear interpolation. Since the payouts are convex in the two state variables, the quadratic interpolation scheme should be effective. Using just three points for each state variable at each node appears to suffice. These results are quite robust to the parameter values for the volatility structure.

[Table 2 Here]

The prices for American interest rate claims, such as options on coupon bonds, display almost identical convergence properties and hence are not reported.

5 Empirical Tests

In this section we provide some preliminary empirical tests on the GGV model. Our goal is to establish if there is support for the one factor GGV model and to establish whether the model can reduce out of sample biases that exist in applying the simple GV model.

We obtained a set of daily caplet data, and zero curves from Bears Stern. The data consist of bid and ask prices of at-the-money caplets with maturities ranging from 1 year to 9 years in increments of 1 year. Each caplet is on a 3 month LIBOR rate. The prices are reported in conventional Black volatility form. Specifically, the Black volatility of a caplet with strike X, and maturity t, on a rate with length Δt years, is the volatility, σ_f, say, that arises under the assumption that the forward rate, corresponding to the caplet period, Δt, has a lognormal distribution. The price is given by:[8]

$$C_0 = \frac{P(0,t)}{1 + f_0[t, t+\Delta t]\Delta t}[f_0[t, t+\Delta_t]N(d_1) - XN(d_2)]$$

where

$$\begin{aligned} d_1 &= \frac{log[f_0[t, t+\Delta t]/X] + t\sigma_f^2/2}{\sigma\sqrt{t}} \\ d_2 &= d_1 - \sigma_f\sqrt{t} \end{aligned}$$

[8]A typical quoted Black volatility of 15% for example, should not be confused with the volatility of the forward rate in our HJM models. The Black model has volatility in a proportional form, whereas the Vasicek models have volatility in absolute form.

To translate Black volatilities into actual prices requires the discount rates for the appropriate maturities. The discount function for each day, computed using the par swap rate curve was also supplied. 10 weeks of data were provided.

In our analysis, we assume all the parameters remain constant over a week. Then we use all $9 \times 5 = 45$ option prices, to infer out the set of estimates that minimize the sum of squared error.[9] We repeat this analysis, separately for each of the 10 successive weeks. Table 3 reports the estimates of the parameters for each of the 10 optimizations for the GV and the GGV models.

[Table 3 Here]

For the GV model, the volatility and mean reversion estimates are fairly stable over the 10 weeks. For the GGV model, in addition to the estimates of the volatility parameters, we also report the forward rate maturity with the maximum volatility. This maturity is consistently close to 1.3 years, and the magnitude of the volatility there is surprisingly stable at a value near 0.015. [10] In all 10 runs of the GGV model, the null hypothesis that the parameter values $a_1 = b_0 = 0$ is rejected at the 5% level of significance. That is, the inclusion of the volatility hump, beyond the usual GV exponentially dampened structure, adds significantly to the model.

Table 4 summarizes the in-sample residuals for the GV and GGV models. In each of the 10 weekly estimations for both models, 5 residuals are available for each maturity. This gives a total of 50 residuals. For each maturity, the number of positive and negative residuals are indicated, as well as their average and standard deviation.

[Table 4 Here]

The table immediately reveals the large biases in the GV model. All the residuals in the first year are negative, while all the residuals in years $2 - 4$ are positive. The large bias continues over all maturities. Table 4 also reports the results for the GGV model. Relative to the GV model, a significant amount of the bias is removed.

The in sample analysis does indicate that a GV model is not capable of fitting a volatility structure that has a hump. Table 5 presents a similar analysis of residuals, this time conducted on out-of-sample data. The estimates of the volatility parameters, derived using the data in a given

[9] We minimized the sum of squared error of the pricing residuals, in dollars, and in Black implied volatility units. In both cases, the sets of results were almost identical. As a result, we use the first objective function, but report all our residuals in Black volatility form. We do this primarily because the typical bid-ask spread on a caplet is of the magnitude of 0.5 Black Vols.

[10] Roughly speaking, this implies that movemnts of $2 \times 1.5 = 3\%$ or more in rates in a year are unlikely.

week, are then used to estimate the option prices for each day of the next week. That is, the volatility parameters are only updated at the end of a week, using full information on the entire week. The model is then not recalibrated until the end of the next week. As a result, for the week, we obtain $9 \times 5 = 45$ out of sample residuals for each maturity. Table 5 reports the number of positive and negative residuals as well as their means and standard deviations for each maturity caplet.

[Table 5 Here]

The results are consistent with the results from the in-sample-residuals. In particular, much of the bias in the GV model is eliminated by the GGV model. The last row of this table reports the number of times (out of 45) that the absolute value of each GGV residual is smaller than the absolute value of the GV residual. The superior performance of the GGV model, especially over the first 5 maturities is evident.

Figures 4a and 4b show the typical plot of the 9 residuals for each of the five consecutive days after the estimation was conducted. As can be seen, the simple exponentially dampened structure for volatilities is not flexible enough to permit pricing to proceed without introducing a large maturity bias.

[Figures 4a and 4b Here]

The above results provide significant evidence that the GGV can explain prices of caplets beyond what is possible with a GV model. Table 6 attempts to establish whether the forecasted GGV prices of caplets are within typical bid ask spreads and whether the forecasts deteriorate over time. For each out-of-sample day, we report the distribution of the $9 \times 9 = 81$ out of sample residuals. For example, consider the one year maturity caplet. Of the 81 forecasts made for the Monday prices, 68 were within 0.25 vols of the actual price, 11 were within 0.5 vols and 2 were larger. The performance of the forecasts did deteriorate somewhat over the next 4 days. However, even if one did not recalibrate the model for 1 week, 71 out of the 81 residuals were within 0.5 vols of the actual prices. Since a typical bid ask spread of a caplet is often between 0.25 and 0.5 vols, residuals of this magnitude are respectable. The results hold true when broken down by caplet maturity.

[Table 6 Here]

Our preliminary empirical results indicate that a significant portion of the bias in the GV model can be explained by a more flexible handling of the volatility structure. Certainly, the results do indicate that the maturity structure of at-the-money caplets can be reasonably well approximated by the GGV model.

Our final analysis was to investigate the magnitude of the biases in using the GGV and GV models for pricing swaptions, which are more sensitive to correlation structures in the forward rates than the caplet prices. Our data set for this study was based on 50 weeks of data, for which our above caplet data was a subset. At each date, 25 swaption contracts, with maturities ranging from one to five years, and forward maturities ranging from one to five years were available. A typical bid-ask spread for a swaption would be about 0.5 of a Black volatility.[11]

Each week the GV and GGV models were calibrated to the data and the resulting residuals for all 25 contracts were stored. Figure 5a and 5b show a series of box and whisker plots of the residuals, for all the 25 contracts spread out across the maturity spectrum. The dashed lines in the exhibit identify typical bid-ask spreads.

[Figures 5a and 5b Here]

The figures clearly illustrate that the majority of residuals fall in the bid-ask spread. More importantly, the analysis shows that the GGV model removes the biases that exist in the short term contract prices produced by the GV model.

6 Conclusion

This paper develops a simple model for pricing interest rate options. Analytical solutions are available for European claims and extremely efficient algorithms exist for the pricing of American claims on the lattice. The interest rate claims are priced in the Heath-Jarrow-Morton paradigm, and hence incorporate full information on the term structure. The volatility structure for forward rates is humped, and includes as a special case the Generalized Vasicek model. The structure of volatilities is captured without using time varying parameters. As a result, the volatility structure is stationary. It is not possible to have a volatility structure with the above properties and at the same time capture the term structure dynamics by a single state variable. It is shown that the full dynamics of the term structure can, however, be captured by a three state Markovian system. As a result,

[11]Under this Black model, the swap rate is assumed to follow a lognormal process. The Black swaption formula is:

$$Swaption_0 = [P(0,t_0) - P(0,t_n)][N(d_1) - N(-d_1)]$$

where t_0 is the expiration date, t_n is the terminal date of the underlying swap, $d_1 = \sigma_s\sqrt{t_0}/2$, and σ_s is the Black volatility.

simple path reconnecting lattices cannot be constructed to price American claims. Nonetheless, we provide extremely efficient lattice based algorithms for pricing claims, which rely on carrying small matrices of information at each node.

Our preliminary empirical analysis provided strong support for the single factor GGV model in favor over the GV model. Moreover, the GGV model produces somewhat stable parameter estimates, and was capable of producing out of sample prices that were consistently within reasonable bid ask spreads. The results indicate that a more thorough empirical study is warranted, where a larger data set is used covering a wider family of contracts.

In our analysis, we assumed the parameters are time invariant constants. It is possible to allow the parameters to be time varying. For example, a_0 and □ could be made functions of time. Then, with $a_1 = b_0 = 0$ the model would reduce to the time varying extended Vasicek model of Hull and White (1990). With minor modifications, our lattice based algorithm can handle these models. Using time varying parameters allows us to price caplets more precisely. In particular, with sufficient free parameters, the in sample sum of squared errors can be reduced to zero. It remains for future work to access whether the out of sample performance of these models provides use beyond the model with no time varying parameters. Since the GGV model, with no time varying parameters is a better specified model than the GV model, it seems sensible to introduce time varying parameters into this model, rather than into the GV model.

It also remains for future work to extend these models to handle a larger class of forward rate volatility structures. As long as the volatility structure is a sum of weighted exponential functions multiplied by maturity dependent polynomials, then a finite state variable representation is possible. When the volatility structure of forward rates belongs to the Ritchken-Sankarasubramanian class the analysis becomes more difficult. Extensions of our lattice procedure to handle humped volatility structures within the extended Ritchken Sankarasubramanian class will be of substantial interest.

There are alternative approaches to calibrating interest rate claim models. Pang (1999) shows how Gaussian random field models can be approximated easily and accurately by multifactor Gaussian HJM models. In addition, he shows that the indirect calibration of these HJM models using Gaussian random field models as an intermediate step is attractive. It would be interesting to compare the effectiveness of hedges constructed following this methodology, with hedges based on the above models.

Appendix

Proof of Proposition 1

By definition of $P(t,T)$ and equation (11), we can write:

$$\begin{aligned} P(t,T) &= e^{-\int_t^T f(t,x)dx} \\ &= \frac{P(0,T)}{P(0,t)} e^{-\int_t^T h_1(t,x)dx - \sum_{i=0}^{2}\int_t^T d_i(t,x)dx W_i(t)} \\ &= A(t,T)e^{-R(t,T)} \end{aligned}$$

where

$$\begin{aligned} A(t,T) &= \frac{P(0,T)}{P(0,t)} e^{-H_1(t,T)} \\ R(t,T) &= \sum_{i=0}^{2}\int_t^T d_i(t,x)dx W_i(t) \\ &= \sum_{i=0}^{2} D_i(t,T)W_i(t) \end{aligned}$$

and

$$\begin{aligned} D_0(t,T) = \int_t^T d_0(t,x)dx &= b_0(T-t) \\ D_1(t,T) = \int_t^T d_1(t,x)dx &= \frac{1}{\square^2}[a_1 + a_0\square - (a_1\square(T-t) + a_0\square + a_1)e^{-\square(T-t)}] \\ D_2(t,T) = \int_t^T d_2(t,x)dx &= \frac{a_1}{\square}[1 - e^{-\square(T-t)}] \end{aligned}$$

$H_1(t,T) = \int_t^T h_1(t,x)dx$ computing this integral yields

$$\begin{aligned} H_1(t,T) \quad &= b_0^2 t T^2/2 + b_0 T(-2a_1 + 2a_1 e^{\square t} - a_0\square + a_0 e^{\square t}\square - a_1\square t \\ &-b_0 e^{\square t}\square^3 t^2)/(e^{\square t}\square^3) \\ &-(5a_1^2 - 5a_1^2 e^{2\square t} + 6a_0a_1\square - 6a_0a_1e^{2\square t}\square + 2a_0^2\square^2 - 2a_0^2 e^{2\square t}\square^2 \\ &+6a_1^2\square t + 4a_0a_1\square^2 t - 16b_0a_1e^{\square t}\square^2 t + 16b_0a_1e^{2\square t}\square^2 t \\ &-8a_0b_0e^{\square t}\square^3 t + 8a_0b_0e^{2\square t}\square^3 t + 2a_1^2\square^2 t^2 - 8b_0a_1e^{\square t}\square^3 t^2 \\ &-4b_0^2e^{2\square t}\square^5 t^3)/(8e^{2\square t}\square^5) \\ &+(-5a_1^2 + 5a_1^2e^{2\square t} - 6a_0a_1\square + 6a_0a_1e^{2\square t}\square - 2a_0^2\square^2 + 2a_0^2e^{2\square t}\square^2 \\ &-6a_1^2e^{2\square t}\square t - 4a_0a_1e^{2\square t}\square^2 t + 2a_1^2e^{2\square t}\square^2 t^2 - 6a_1^2\square T + 6a_1^2e^{2\square t}\square T \end{aligned}$$

$$
\begin{aligned}
&-4a_0a_1\Box^2T + 4a_0a_1e^{2\Box t}\Box^2T - 4a_1^2e^{2\Box t}\Box^2tT - 2a_1^2\Box^2T^2 \\
&+2a_1^2e^{2\Box t}\Box^2T^2)/(8e^{2\Box T}\Box^5) \\
&+(5a_1^2 - 5a_1^2e^{2\Box t} + 6a_0a_1\Box - 6a_0a_1e^{2\Box t}\Box + 2a_0^2\Box^2 - 2a_0^2e^{2\Box t}\Box^2 \\
&+3a_1^2\Box t + 3a_1^2e^{2\Box t}\Box t + 2a_0a_1\Box^2t - 8b_0a_1e^{\Box t}\Box^2t + 2a_0a_1e^{2\Box t}\Box^2t \\
&+8b_0a_1e^{2\Box t}\Box^2t - 4a_0b_0e^{\Box t}\Box^3t + 4a_0b_0e^{2\Box t}\Box^3t - 4b_0a_1e^{2\Box t}\Box^3t^2 \\
&+3a_1^2\Box T - 3a_1^2e^{2\Box t}\Box T + 2a_0a_1\Box^2T + 8b_0a_1e^{\Box t}\Box^2T - 2a_0a_1e^{2\Box t}\Box^2T \\
&-8b_0a_1e^{2\Box t}\Box^2T + 4a_0b_0e^{\Box t}\Box^3T - 4a_0b_0e^{2\Box t}\Box^3T + 2a_1^2\Box^2tT \\
&-4b_0a_1e^{\Box t}\Box^3tT + 8b_0a_1e^{2\Box t}\Box^3tT + 4b_0a_1e^{\Box t}\Box^3T^2 - 4b_0a_1e^{2\Box t}\Box^3T^2) \\
&/(4e^{\Box(t+T)}\Box^5)
\end{aligned}
$$

Proof of Proposition 2

By definition of call option which expires at date t:

$$
\begin{aligned}
C(t) &= Max[P(t,T) - X, 0] \\
&= Max[A(t,T)e^{-R(t,T)} - X, 0]
\end{aligned}
$$

The expected payoff under the FRA measure at date t is given by:

$$
E_0[C(t)] = A(t,T)e^{\gamma^2(t,T)/2}N(d_1) - XN(d_2)
$$

where

$$
\begin{aligned}
d_1 &= \frac{log(A(t,T)/X) + \gamma^2(t,T)}{\gamma(t,T)} \\
d_2 &= d_1 - \gamma(t,T)
\end{aligned}
$$

and $\gamma^2(t,T) = Var(R(t,T))$. From equation (4), we can write:

$$
\begin{aligned}
C(0) &= P(0,t)E_0[C(t)] \\
&= P(0,t)A(t,T)e^{\gamma^2(t,T)/2}N(d_1) - P(0,t)XN(d_2) \\
&= P(0,T)e^{-H_1(t,T)+\gamma^2(t,T)/2}N(d_1) - P(0,t)XN(d_2) \\
&= P(0,T)N(d_1) - P(0,t)XN(d_2)
\end{aligned}
$$

Note that $H_1(t,T) = \gamma^2(t,T)/2$.

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Table 1
Convergence Rate of Options With
Linear and Quadratic Interpolations*

	Linear						Quadratic
N	**k=2**	**k=3**	**k=4**	**k=10**	**k=20**	**k=50**	**k=3**
2	7.256	7.256	7.256	7.256	7.256	7.256	7.256
3	8.768	8.768	8.768	8.768	8.768	8.768	8.768
4	7.664	7.651	7.649	7.647	7.647	7.647	7.649
5	8.470	8.470	8.470	8.470	8.470	8.470	8.470
10	7.932	7.908	7.897	7.890	7.889	7.889	7.892
25	8.118	8.118	8.118	8.118	8.118	8.118	8.118
50	8.081	8.046	8.038	8.022	8.016	8.014	8.014
100	8.101	8.063	8.057	8.038	8.032	8.028	8.026
200	8.126	8.078	8.070	8.046	8.040	8.035	8.032
500	8.272	8.153	8.113	8.061	8.047	8.039	8.034
1000	8.506	8.270	8.191	8.086	8.059	8.043	8.034
Exact	8.033	8.033	8.033	8.033	8.033	8.033	8.033

Table 1 shows the convergence rate of a call option as the number of time partitions, n, increases. The maturity of the option is 6 months. The underlying bond is a two year bond. The strike price is set equal to the current forward price, for delivery in 6 months. The table shows the convergence rate for various values of k , when linear interpolation procedures are used, and for a quadratic interpolation scheme. The initial term structure is given by:

$$f(0,t) = 0.07 - 0.02e^{-0.18t} .$$

The case parameters for the volatility structure are:

$$\boldsymbol{k} = 0.1,\ a_0 = 0.02,\ a_1 = 0,\ b_0 = 0.003 .$$

In this case there are two state variables in the model.

Table 2
Convergence Rate of Options With
Linear and Quadratic Interpolations*

	Linear						Quadratic
N	**k=2**	**k=3**	**k=4**	**k=10**	**k=20**	**k=50**	**k=3**
2	7.925	7.925	7.925	7.925	7.925	7.925	7.925
3	9.611	9.611	9.611	9.611	9.611	9.611	9.611
4	8.432	8.400	8.396	8.396	8.396	8.396	8.395
5	9.314	9.314	9.314	9.314	9.314	9.314	9.314
10	8.761	8.730	8.713	8.695	8.695	8.695	8.698
25	8.963	8.962	8.962	8.962	8.962	8.962	8.962
50	8.941	8.911	8.897	8.867	8.855	8.850	8.852
100	8.987	8.940	8.921	8.890	8.878	8.868	8.868
200	9.115	8.993	8.957	8.905	8.890	8.880	8.874
500	9.466	9.171	9.073	8.942	8.907	8.888	8.877
1000	10.031	9.459	9.266	9.007	8.938	8.900	8.877
Exact	8.876	8.876	8.876	8.876	8.876	8.876	8.876

*Table 2 shows the convergence rate of a call option as the number of time partitions, n, increases. The maturity of the option is 6 months. The underlying bond is a two year bond. The strike price is set equal to the current forward price, for delivery in 6 months. The table shows the convergence rate for various values of k , when linear interpolation procedures are used, and for a quadratic interpolation scheme. The initial term structure is given by:

$$f(0,t) = 0.07 - 0.02e^{-0.18t} .$$

The case parameters for the volatility structure are:

$$\boldsymbol{k} = 0.1,\ a_0 = 0.02,\ a_1 = 0.0025,\ b_0 = 0.003 .$$

In this case there are three state variables in the model

Table 3
Weekly Estimates of Parameters*

	GV Estimates		GGV Estimates					
week	a_0	κ	a_0	a_1	a_2	κ	Hump	Max. Vol.
1	0.0133	0.0346	-0.0221	0.0410	0.0100	1.3087	1.3034	0.0157
2	0.0129	0.0282	-0.0203	0.0363	0.0101	1.2647	1.3497	0.0153
3	0.0130	0.0263	-0.0230	0.0401	0.0104	1.3394	1.3193	0.0155
4	0.0131	0.0239	-0.0230	0.0400	0.0107	1.3679	1.3054	0.0156
5	0.0131	0.0226	-0.0328	0.0535	0.0109	1.5416	1.2627	0.0158
6	0.0131	0.0201	-0.0308	0.0506	0.0111	1.5326	1.2612	0.0158
7	0.0132	0.0259	-0.0273	0.0470	0.0107	1.4548	1.2683	0.0158
8	0.0130	0.0265	-0.0271	0.0468	0.0105	1.4697	1.2579	0.0156
9	0.0130	0.0226	-0.0232	0.0387	0.0105	1.3007	1.3695	0.0155
10	0.0131	0.0231	-0.0216	0.0373	0.0106	1.3095	1.3439	0.0155

*Table 3 shows the implied estimates of the forward rate volatility parameters in each of 10 successive weeks. Each estimate is based on 45 caplet prices spanning the maturity spectrum.

Table 4
In-Sample Residual Analysis*

Model	Caplet Maturity	1	2	3	4	5	6	7	8	9
GV	positive	0*	50*	50*	50*	48*	17	3*	2*	14*
	negative	50	0	0	0	2	33	47	48	36
	average	-3.330*	0.763*	0.684*	0.455*	0.269*	-0.087*	-0.178*	-0.206*	-0.094*
	s.d.	0.290	0.165	0.164	0.155	0.150	0.200	0.132	0.140	0.141
GGV	positive	23	37*	9*	32	42*	21	22	19	32
	negative	27	13	41	18	8	29	28	31	18
	average	-0.007	0.055*	-0.124*	0.021	0.137*	-0.038	-0.041*	-0.046*	0.043*
	s.d.	0.177	0.170	0.170	0.150	0.144	0.194	0.132	0.135	0.134

*Table 4 shows the in-sample residuals by caplet maturity. For example, the GV model produce 48 positive residuals and two negative residuals for the 5 year maturity caplet. The starred values indicate the proportions (means) that were significantly different from one half (zero). All tests were done at 0.5% level of significance.

Table 5
Out-Of-Sample Residual Analysis[+]

Model	Maturity	1	2	3	4	5	6	7	8	9
GV	positive	0*	44*	44*	43*	38*	24	14*	17	24
	negative	45	1	1	2	7	21	31	28	21
	average	-3.363*	0.760*	0.700*	0.480*	0.298*	-0.044	-0.130*	-0.138*	-0.030
	s.d.	0.347	0.247	0.250	0.242	0.244	0.304	0.236	0.239	0.246
GGV	positive	25	26	16	30	35*	26	26	27	30
	negative	20	19	29	15	10	19	19	18	15
	Average	-0.037	0.038	-0.111*	0.048	0.168*	0.007	0.008	0.017	0.101*
	s.d.	0.406	0.275	0.247	0.227	0.231	0.301	0.234	0.239	0.252
GV Wins[++]		0	3	6	6	9	23	20	21	27
GGV Wins		45*	42*	39*	39*	36*	22	25	24	18
Number of Trials		45	45	45	45	45	45	45	45	45

[+] Table 5 shows the out-of-sample residuals by caplet maturity. For example, the GV model produces 38 positive residuals and 7 negative residuals for the 5 year caplet. The starred values indicate the proportions (means) that were significantly different from one half (zero). All test were done at 0.5% level of significance.

[++] GV wins if the absolute value of the residual is smaller than the absolute value of the corresponding GGV model. Otherwise GGV wins.

Figure 1
Illustration of Volatility Hump*

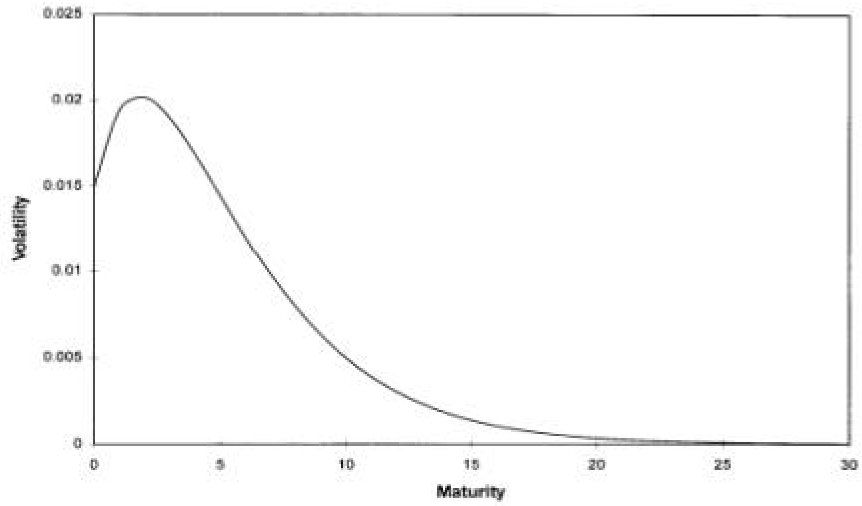
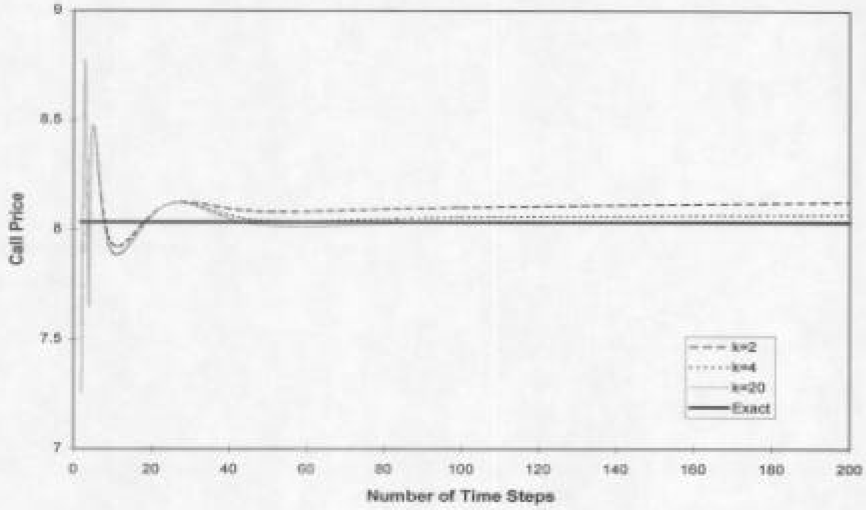


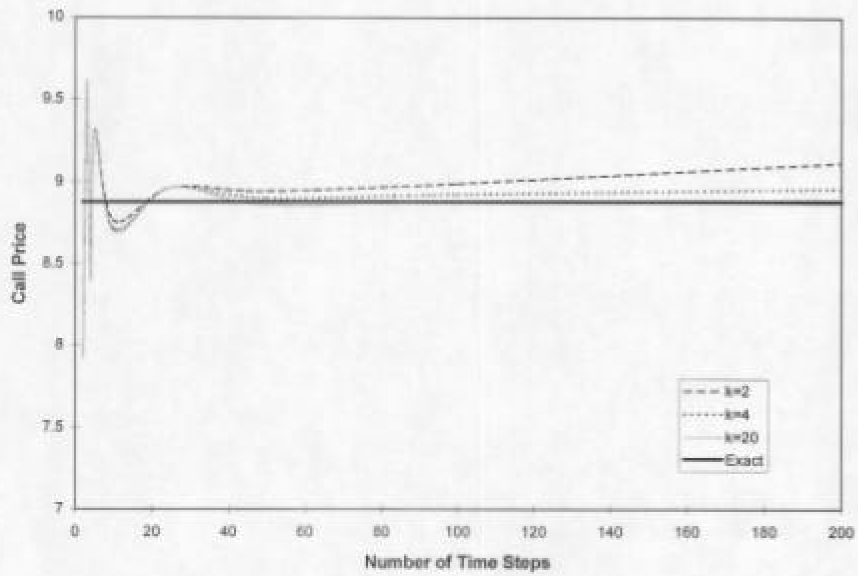
*Figure 1 shows a typical volatility structure that can be established using equation (9). In this figure $b_0 = 0$ so the volatility for long term forward rates eventually decays to zero.

Figure 2
Convergence of Option Prices on The Lattice*



*The volatility structure for this figure is detailed in Table 1. There are two state variables for this volatility structure, so at each node in the lattice, there is a vector of prices. The size of the vector is k. The figure shows the convergence rate of prices for three different k values. The top dashed line corresponds to the case where k = 2. The middle dashed line corresponds to the case where k = 4, and the almost flat solid line corresponds to the case where k = 20. The example illustrates that accurate prices can be obtained when k is reasonably large. The option is a six month European call option on a two year bond. The exact specifications of the contract are discussed in Table 1.

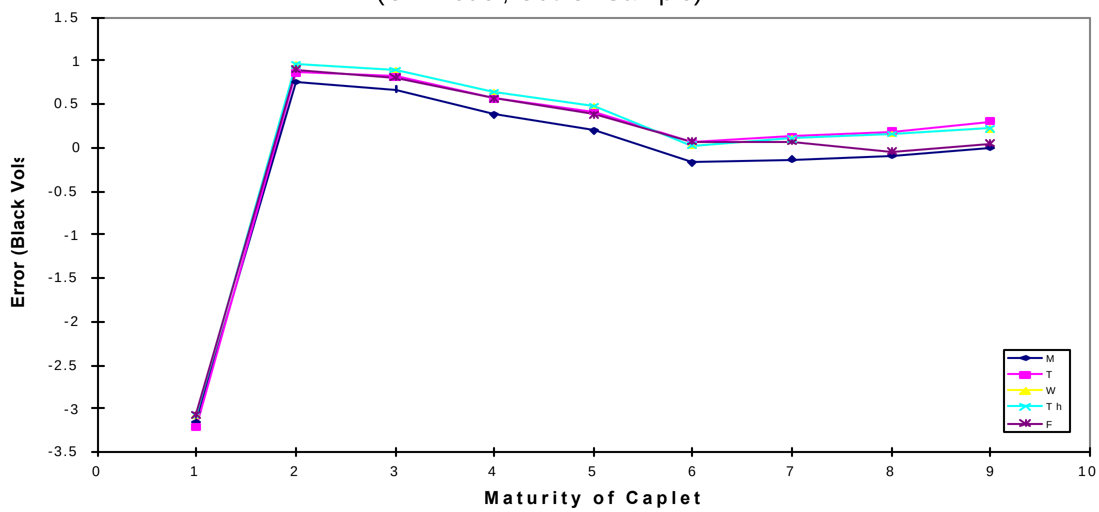
Figure 3
Convergence of Option Prices on The Lattice*



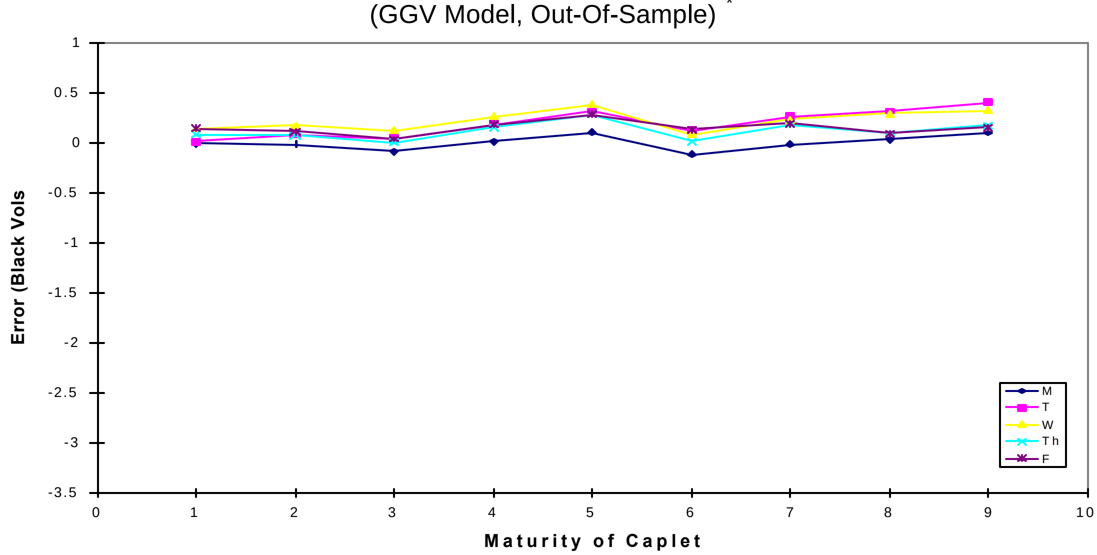
*The volatility structure for this figure is detailed in Table 2. There are three state variables for this volatility structure, so at each node in the lattice, there is a matrix of prices. The size of the matrix is k × k. The figure shows the convergence rate of prices for three different k values. The top dashed line corresponds to the case where k = 2. The middle dashed line corresponds to the case where k = 4, and the almost flat solid line corresponds to the case where k = 20. The example illustrates that accurate prices can be obtained when k is reasonably large. The option is a six month European call option on a two year bond. The exact specifications of the contract are discussed in Table 2.

Figure 4
Residual Plots

(a) Plot of Daily Residuals For A Given Week
(GV Model, Out-Of-Sample) *



(b) Plot of Daily Residuals For A Given Week
(GGV Model, Out-Of-Sample) *



*This figure shows the residual (in Black vol form) for each caplet maturity for each day in a typical week. Figure 4a shows the residuals for the GV model while figure 4b shows the residuals for the GGV model.

Figure 5
Boxplots Of Residuals From In-sample Fits*

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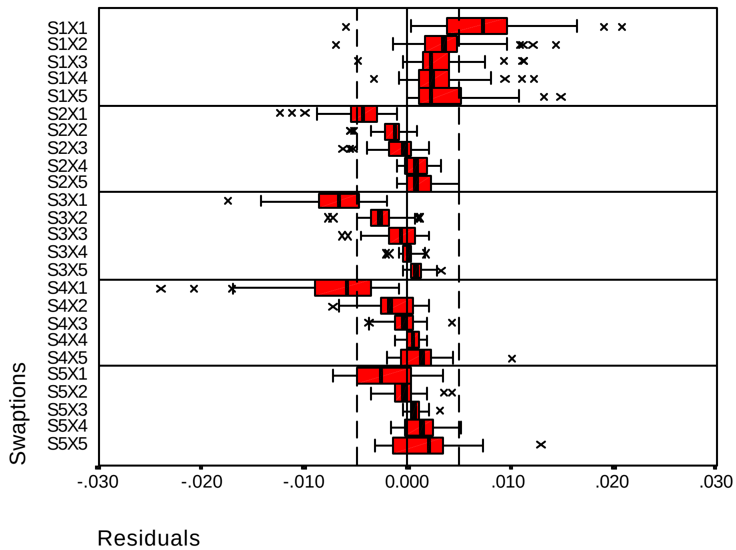
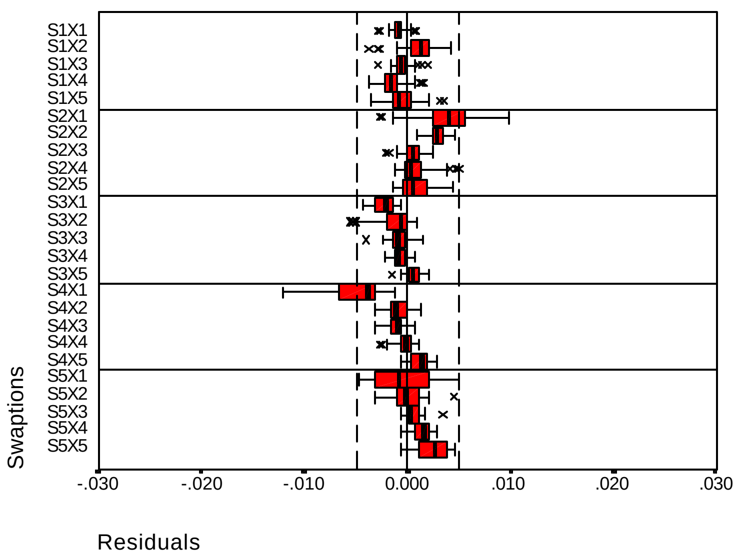
(a) One-factor GV, in-sample



(b) One-factor GGV, in-sample



* Figure 5 show the boxplots of residuals from the in-sample fits for one-factor GV, and a one-factor GGV respectively. Each plot is based on the median, quartiles, and outliers of 50 swaption contracts. The box represents the interquartile range that contains the 50% of values. The whiskers are lines that extend from the box to the highest and lowest values, excluding outliers. A line across the box indicates the median. The “x” means outliers. Also, we plot the bid-ask spreads with dashed lines. In most cases, the boxes fall in the bid-ask spreads. A S2XS1 contract refers to a two year swaption, on a one year swap.