Chapter 6
Arbitrage Relationships for Call and Put Options

Recall that a risk-free *arbitrage opportunity* arises when an investment is identified that requires no initial outlays yet guarantees nonnegative payoffs in the future. Such opportunities do not last long, as astute investors soon alter the demand and supply factors, causing prices to adjust so that these opportunities are closed off. In this chapter we shall use simple arbitrage arguments to obtain some basic boundary conditions for call and put options. The beauty of the pricing relationships derives from the fact that they require no assumptions on the statistical process driving security prices. Also, no severe assumptions are made concerning the risk behavior of investors. The simple requirement is that investors like more money than less. This chapter also investigates conditions under which it is more appropriate to exercise options than to sell them. The final section explores some fundamental pricing relationships that exist between put and call options.

The primary objectives of this chapter are the following:

- To derive boundary prices for call and put option prices from arbitrage arguments;
- To explain when exercising options is not appropriate;
- To understand how mispriced options can be traded to lock in arbitrage free profits; and
- To describe pricing relationships between put and call options.

Throughout this chapter we shall adopt the following notation. The current time is $t = 0$. The option expires at date T, where T is expressed in years. The time to the j^{th}ex-dividend date is t_j, and the size of dividend declared at ex-dividend date t_j is d_j. We are interested only in ex-dividend dates prior to expiration. In most cases, the number of dates in the interval is less than three. Throughout this chapter we assume the risk free rate, r, is constant. Let $B(s,t)$ be the price at time s of a riskless pure discount bond that pays out \$1 at time t. Then

$$B(s,t) = e^{-r(t-s)} \leq 1$$

and with $s = 0$:

$$B(0,t) = e^{-rt}$$

Clearly, discount bond prices decrease with maturity. That is, for $t > s$

$$B(0,t) < B(0,s) < 1$$

If 1 dollar is invested in the risk free asset (bank) at time s, this amount of money will grow continuously at rate r, and at time t the value will be $1e^{r(t-s)}$. Let $G(s,t) = e^{r(t-s)}$ represent the account value. Clearly, $G(s,t) = 1/B(s,t)$.

Finally, as discussed in chapter 4, a call option that can only be exercised at the expiration date (and not before) is called a European option. An American option must have a value at least as great as a European option, since the former has all the properties of the latter plus the additional early exercise feature. This characteristic is used in deriving several properties in this chapter.

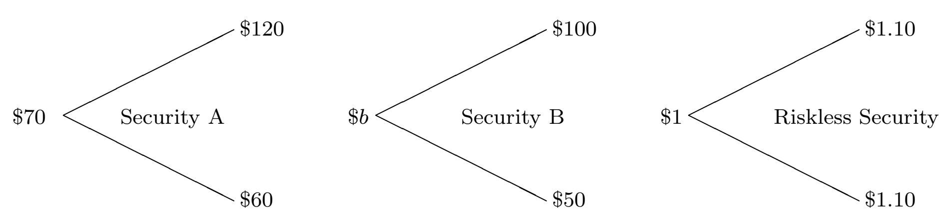
Riskless Arbitrage and the Law of One Price

This chapter frequently uses the argument that there are no riskless arbitrage opportunities. Below we provide some illustrative examples.

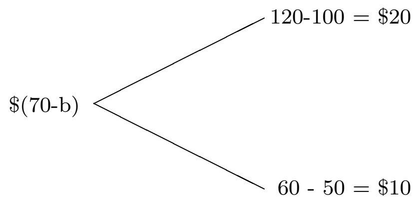
Example

(i) Consider the following investment opportunities:

- For \$70, an investor can buy (sell) a share of A, which, at the end of the period, will either appreciate to \$120 or depreciate to \$60, depending on whether the economy "booms" or not.
- For \$$b$, an investor can buy (sell) a share of B that will either appreciate to \$100 or depreciate to \$50, depending on the same economic factors.
- A riskless investment where each dollar grows to \$1.10.

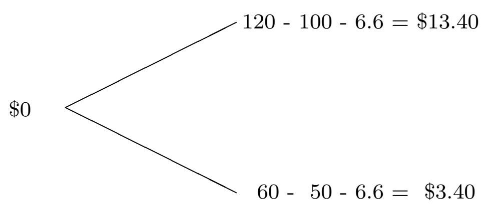


What is the maximum price b can take?



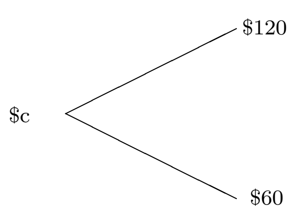
To answer this question, first consider a portfolio consisting of one share of A and a short position of one share in B. The payouts of this portfolio are shown below:

Clearly, all investors would prefer the final dollar payout of this portfolio to a certain payout of \$10. To avoid a possible "free lunch", the present value of this portfolio must exceed the present value of \$10. That is, $(70 - b) > 10/1.10 = \$9.09$, or $b < \$60.91$. If, for example, $b = \$64$, then a "free lunch" would exist. Specifically, an investor could establish a zero initial investment position by selling one share of B for \$64, borrowing \$6, and using the total proceeds to purchase one share of A. The \$6 debt will grow to $6 \times 1.1 = \$6.60$ in one year. The final payouts of this strategy are shown below:



To avoid this free lunch, the price of a share of B must satisfy $b < \$60.91$.

Assume a third investment, C, provided the following payout:



To avoid riskless arbitrage, c must equal \$70. To see this, note that this investment has identical payouts to A. If C exceeded \$70, investors would buy A and sell C to lock into profits. Conversely, if C was lower than \$70, investors would buy C and sell A to lock into profits.

The *law of one price* states that if two securities produce identical payouts in all future states, then, to avoid riskless arbitrage, their current prices must be the same. We shall use this law frequently in this and later chapters.

Call Pricing Relationships

In this section we establish bounds on call options and we investigate conditions under which it may be optimal to exercise the call option. Our first property provides a bound for the price of a call option when the underlying stock pays no dividends over the lifetime of the option.

Property 1

If there are no dividends prior to expiration, then to prevent arbitrage opportunities, the call price should never fall below the maximum of zero or the stock price minus the present value of the strike. That is,

$$C_0 \geq Max[0, S_0 - XB(0,T)]$$

Proof: Consider two portfolios, A and B. A contains one European call option and X pure discount bonds with a face value of \$1 each and maturity T. B contains a long position in the stock.

Exhibit 1 llustrates the prices of the two portfolios at the expiration date of the option. Note that the future value of portfolio A is never lower than the future value of portfolio B.

Exhibit 1: Bounding Call Prices

Portfolio	Current Value	$S_T < X$	$S_T \geq X$
A	$C_0 + XB(0,T)$	$0 + X$	$(S_T - X) + X$
B	S_0	S_T	S_T
		$V_A(T) > V_B(T)$	$V_A(T) = V_B(T)$

If an investor bought portfolio A and sold portfolio B, then, at the expiration date, the combined portfolio, P, would have value $V_p(T)$, given by $V_p(T) = V_A(T) - V_B(T)$, where $V_A(T)$ and $V_B(T)$ define the values of the portfolios A and B at time T.

If the call option expired in the money, then $V_A(T) = V_B(T)$ and, hence, $V_p(T) = 0$. However, if the call expired worthless, then, $V_p(T) = V_A(T) - V_B(T) \geq 0$. The portfolio, P, thus, can never lose money and has a chance of making money. Let us now consider the initial cost of the portfolio, $V_p(0)$. Since this portfolio has a nonnegative terminal value, it must be worth a nonnegative amount now. Hence

$$V_p(0) = V_A(0) - V_B(0) \geq 0$$

Equivalently, $V_A(0) \geq V_B(0)$. Substituting for $V_A(0)$ and $V_B(0)$, leads to

$$C_0 \geq S_0 - XB(0,T)$$

Since a European call option must have value no less than $S_0 - XB(0,T)$, so must an American option. Of course, since call options offer the holder the right to purchase securities at a particular price, this right must have some value. Hence, $C_0 \geq 0$. The result then follows.

Example

Consider a stock currently priced at \$55, with a three-month \$50 strike price call available. Assume no dividends occur prior to expiration, and the riskless rate, r, is 12%. The lower bound on the price is given by the following:

$$C_0 \geq \text{ Max}[0, S_0 - XB(0,T)] = \text{ Max}[0, 55 - 50e^{-0.12(3/12)}] = \$6.48$$

We now establish bounds on the call price when the underlying security pays a dividend.

Property 2

If a stock pays a single dividend of size d_1 at time t_1, then to prevent riskless arbitrage, the call price should never fall below

$$Max[C_1, C_2]$$

where

$$\begin{aligned} C_1 &= Max[0, S_0 - XB(0,t_1)] \\ C_2 &= Max[0, S_0 - d_1B(0,t_1) - XB(0,T)] \end{aligned}$$

Proof: C_1, and C_2 correspond to values from specific trading strategies.

Strategy 1: Exercise the Call Option Just Prior to Ex-Dividend Date

Assume the option is exercised just prior to the ex-dividend date. Then, by using the same argument used to prove Property 1, we see that the call price must exceed the stock price less the present value of the strike.

Strategy 2: Exercise the Call Option at Expiration

If the dividend is sacrificed and the option held to expiration, the call value must exceed C_2. To prove this result, consider two portfolios, A and B. A contains d_1 bonds that pay out $\$d_1$ at time t_1 and X bonds that pay out X dollars at time T. In addition, a call option is held. All dividends received at time t_1 are invested in the risk-free asset. Portfolio B contains a long position in the stock. Exhibit 2 illustrates the payoffs that occur if the strategy of exercising the call at the expiration date is followed.

Exhibit 2: Bounding Call Prices with Dividends

Portfolio	Current Value	$S_T < X$	$S_T \geq X$
A	$C_0 + XB(0,T) + d_1 B(0,t_1)$	$X + d_1 G(t_1,T)$	$S_T + d_1 G(t_1,T)$
B	S_0	$S_T + d_1 G(t_1,T)$	$S_T + d_1 G(t_1,T)$
		$V_A(T) \geq V_B(T)$	$V_A(T) = V_B(T)$

Note that in portfolio A, the d_1 dollars received at time t_1 are reinvested in the riskless security and thus grow to $d_1 G(t_1,T)$ at expiration.

Since $V_A(T) \geq V_B(T)$, it must follow that,to prevent arbitrage opportunities, $V_A(0) \geq V_B(0)$. Hence, $C_0 \geq C_2$.

Since at the current time the optimal strategy is unknown, the actual call value should exceed the payoffs obtained under these two strategies. Hence, $C_0 \geq \text{Max}(C_1, C_2)$. This completes the proof.

Example

Reconsider the previous problem, but now assume a dividend of \$5 is paid after one month. The lower bound on the call price is given by $\text{Max}(C_1, C_2)$, where

$$\begin{aligned} C_1 &= Max[0, S_0 - Xe^{-rt_1}] = Max(0, 55 - 49.5) = \$5.50, \\ C_2 &= Max[0, S_0 - d_1 e^{-rt_1} - Xe^{-rT}] = Max(0, 55 - 4.95 - 48.52) = \$1.53. \end{aligned}$$

Hence, $C_0 \geq \$5.50$. Note that the effect of dividends has been to lower the lower bound of the call option.

Example

An April 50 call option on a stock, XYZ, currently priced at \$66.25 currently trades at \$16.75. The current interest rate is 8.30%. A dividend of \$0.75 is due, with the ex-dividend date being 58 days away. The time to expiration is 136 days. To avoid riskless arbitrage opportunities a lower bound on the call price is

$$\begin{aligned} C_0 &\geq S_0 - B(0,T)X - B(0,t_1)d_1 \\ &= 66.25 - e^{-0.083\times(136/365)}50 - e^{-0.083\times(58/365)}0.75 \\ &= \$17.03 \end{aligned}$$

Since the actual price is \$16.75, an arbitrage opportunity exists. Specifically, since the call is underpriced, buy the call, short 1 share of stock, invest \$48.477 for time T, and invest \$0.740 for time t_1, at the riskless rate. The cash flow at time zero and at time T is given below:

Exhibit 3: Cash Flows Illustrating Arbitrage

	Initial Cash Flow	Final Cash Flow at Date T	
		$S_T < 50$	$S_T \geq 50$
sell stock	66.25	$-S_T$	-S_T
buy call	-16.75	0	$S_T - 50$
invest at risk free rate for time T	-4847	50	50
invest at risk free rate for time t_1 (used to pay dividend)	-0.74		
	0.28	$V(T) = 50 - S_T \geq 0$	$V(T) = 0$

Note that if the dividend was uncertain but bounds on its value could be established such that $d_{\min} < d_1 < d_{\max}$ then a lower bound on the call value could be obtained by using $d_{\max}$. Note also that if there are two or more dividends prior to expiration, then lower bounds can be obtained by simple extensions to Property 2. For example, if two certain dividends d_1 and d_2 occur, then $C_0 \geq \text{Max}(C_1, C_2, C_3, C_4)$ where

$$\begin{aligned} C_1 &= Max[0, S_0 - XB(0,t_1)] \\ C_2 &= Max[0, S_0 - d_1B(0,t_1) - XB(0,t_2)] \\ C_3 &= Max[0, S_0 - d_1B(0,t_1) - d_2B(0,t_2) - XB(0,T)]. \end{aligned}$$

Optimal Exercise Policy For Call Options

The previous property established bounds on the call prices by considering the effect of specific exercise strategies: namely, exercising immediately, just prior to an ex- dividend date, or at expiration. However, other exercising times are possible. In this section we show that if exercising is ever appropriate for call options, it should be done at expiration or immediately prior to an ex-dividend date.

Property 3

(i) The early exercise of a call option on a stock that pays no dividends prior to expiration is never optimal.
(ii) For such a stock, the price of an American call option equals the price of an otherwise identical European call.

Proof: To prove this result we must show that for a stock that pays no dividends over the lifetime of the option, the value of a call option unexercised is always equal to or greater than the value of the option exercised.

Let t_p be any time point prior to expiration. From Property 1, the lower bound on the call price at time t_p is the stock price, $S(t_p)$, less the present value of the strike, $XB(t_p, T)$. Hence, $C(t_p) \geq \text{Max}[0, S(t_p) - XB(t_p, T)]$. Note that the right hand side of the equation exceeds the intrinsic value of the option, $S(t_p)$ - X. Hence, early exercise is not optimal.

From Property 3 we see that early exercise of an American call option on a stock that pays no dividends over the lifetime of the option is never appropriate. Therefore, the value of the right to exercise the option prior to expiration must be zero. Thus, an American call option must have the same value as a European call.

Property 4

If a stock pays dividends over the lifetime of the option, then the American call option may be worth more than a European call option.

Proof: To prove this result, we shall show that just prior to an ex-dividend date there may be an incentive to exercise early. To see this, consider an extreme case in which a firm pays all its assets as cash dividends. Clearly, any in-the-money call options should be exercised prior to the ex-dividend date, since after the date the call value will be zero. Note that if the option was European, its value before the ex-dividend date would be zero, whereas the American option would have positive value.

Property 5

The exercise of a call option is optimal only at expiration, or possibly at the instant prior to the ex dividend date.

Proof: Exercising of call options is appropriate only if the value of the call unexercised falls below the intrinsic value. Property 5 says that the only times the call value unexercised may equal or fall below its intrinsic value are immediately prior to an ex-dividend date and at expiration. To prove this property let t_p be any possible exercising date. We have already seen that if t_p is set such that there are no more dividends prior to expiration, early exercise is not appropriate. If t_p is set before the ex-dividend date t_1, early exercise is again not appropriate. To see this, recall that a lower bound of a call price at time t_p is given by $C \geq C_1$, where $C_1 = Max[0, S(t_p) - XB(t_p, t_1)]$ is the bound obtained by delaying exercise to immediately prior to the ex-dividend date. Moreover, for $t_p < t_1$, C_1 can never be equal to its intrinsic value. At time t_1 the lower bound can be attained and early exercise may be optimal.

Dividend and Income Yield Analysis for Call Options

Property 5 states that if early exercise occurs, it should occur just prior to an ex-dividend date. The decision to exercise involves a trade-off between dividend income and interest income. To illustrate this more precisely, consider a stock that pays a single dividend prior to expiration. Just prior to the ex-dividend date the value of an in-the-money call, if exercised, is its intrinsic value, $S(t_1)$-X. Just after the ex- dividend date, the stock price falls to $S(t_1) - d_1$. Since there are no more dividends prior to expiration, from Property 1 we have:

$$C(t_1) \geq (S(t_1) - d_1) - XB(t_1, T)$$

Hence, early exercise will not be optimal if:

$$(S(t_1) - d_1) - XB(t_1, T) \geq S(t_1) - X$$

or

$$d_1 \leq X[1 - B(t_1, T)] \quad (1)$$

The right hand side of the above equation equals the present value of the interest generated from the strike price over the time period $[t_1, T]$, viewed from time t_1. The above equation states that if the dividend is less then the present value of this interest, then early exercise is not optimal. This leads to the following property.

Property 6

(i) Early exercise of the American call option on a stock that pays a single dividend prior to expiration is never optimal if the size of the dividend is less than the present value of interest earned on the strike from the ex dividend date to expiration.

(ii) Early exercise of an American call option on a stock that pays multiple dividends prior to expiration is never optimal if at each ex dividend date, the present value of all future dividends till expiration is less than the present value of interest earned on the strike, from the ex dividend date to expiration.

Example

A stock is priced at \$55. It pays a \$5.00 in one month. Interest rates are 12% per year. A three month call option with strike \$50 has a lower bound given by:

$$\begin{aligned} C_0 &\geq Max(C_1, C_2) \\ &= Max(5.50, 1.53) = \$5.50. \end{aligned}$$

Early exercise of this option will not be appropriate if the size of the dividend ($d_1 = \$5$) is lower than the foregone interest, f, which is given by:

$$f = X[1 - e^{-r(T-t_1})] = 50[1 - e^{-0.12\times(2/12)}] = \$0.99.$$

Since $d_1 > f$, exercise just prior to the ex dividend date may be appropriate.

Note if there are no dividends, d_1 is zero, the early exercise condition in equation (1) is satisfied and premature exercising is not optimal. Moreover, from equation (1), the option should not be exercised if the strike price exceeds the value X^* where

$$X^* = d_1/[1 - B(t_1, T)].$$

If equation (1) is satisfied, early exercise of the option is never optimal. If the equation is not satisfied, then early exercise may be optimal. Of course, the benefit from exercising is that the intrinsic value is received prior to the stock price dropping by the size of the dividend. This benefit is partially offset by the loss of the time premium of the option. If at the ex- dividend date the stock price is sufficiently high, then the time premium is very small, and if equation (1) does not hold, early exercise is quite likely. We defer further details of this point to Chapter 9, where explicit models for the time premium are established,and rules are established for precise timing of exercise.

Put Pricing Relationships

In this section some arbitrage restrictions for American put options are derived. Unlike American call options, early exercise of American put options may be optimal even if the underlying stock pays no dividends. Before considering early exercise policies, we first establish some pricing bounds. As with Property 2 for calls, we obtain pricing bounds by considering specific exercise strategies for put options.

Property 7

Consider a stock that pays a single dividend of size d_1 at time t_1. An American put option must satisfy the following:

$$P_0 \geq Max[P_1, P_2]$$

where

$$\begin{aligned} P_1 &= Max[0, X - S_0] \\ P_2 &= Max[0, (X + d_1)B(0, t_1) - S_0] \end{aligned}$$

Proof. To obtain these bounds we consider two strategies. The first is to exercise immediately; the second strategy is to exercise just after the ex dividend date. Other strategies, such as exercising just before the ex-dividend date, or exercising at expiration, lead to weaker bounds.

Strategy 1: Exercise Immediately

Since a put option gives the holder the right (but not the obligation) to sell stock at the strike price, the put option must have nonnegative value. Further, if exercised immediately, its intrinsic value is obtained with the loss of a possible time premium. Hence, $P_0 \geq P_1$.

Strategy 2: Exercise Immediately After the Ex-dividend Date Now consider the strategy of exercising the put option just after the ex-dividend date, t_1. Then $P_0 \geq P_2$. To see this, consider two portfolios, A and B, where A consists of the put and B consists of a short position in the stock together with $(X + d_1)$ bonds that mature at time t_1. Exhibit 4 shows their values at time t_1, given that the put is exercised.

Exhibit 4: Bounding Put Options

Portfolio	Current Value	Value at date t_1	
		$S(t_1) < X$	$S(t_1 \geq X$
A	P_0	$X - S(t_1$	0
B	$(d_1 + X)B(0, t_1) - S_0$	$X - S(t_1)$	0
		$V_A(T) = V_B(T)$	$V_A(T) > V_B(T)$

To avoid risk-free arbitrage opportunities, it must follow that the current value of portfolio A exceeds that of B. That is,

$$P_0 \geq (d_1 + X)B(0, t_1) - S_0 = P_2$$

We leave it as an exercise to show that the strategy of exercising the put at expiration or prior to the ex-dividend date also leads to weaker bounds. Intuitively, exercising the put prior to an ex-dividend date is not sensible, since immediately after the ex- dividend date the stock price will be lower.

Example

Consider a stock priced at $55, with a three-month put option with strike 60 available. Assume the riskless rate r is 12 percent and a dividend of size $2 is due in one month. Then, we have

$$\begin{aligned} P_0 &\geq Max(P_1, P_2) \\ P_1 &= Max[0, X - S_0] = Max[0, 60 - 55] = 5 \\ P_2 &= Max[0, (X + d_1)B(0, t_1) - S_0] = Max[0, 62e^{-0.12\times(1/12)} - 55] = 6.383 \end{aligned}$$

Example

An April 15 put on XYZ, currently trading at $11.5, is selling for $3 $\frac{1}{2}$. The time to expiration is 107 days and the ex-dividend date is in 57 days. The size of the dividend is $0.60. The current borrowing and lending rate is 9.5%.

From the above analysis a lower bound on the put price is given by

$$\begin{aligned} P_0 &\geq (X + d_1)B(0, t_1) - S_0 \\ &= (15 + 0.60)e^{-0.09\times(57/365)} - 11.50 \\ &= \$3.88 \end{aligned}$$

Since the actual put price is $3.5, this condition is violated and risk free arbitrage is possible. Specifically the put should be purchased, and portfolio B (see Exhibit 4) should be sold. In this case selling portfolio B implies buying the stock and borrowing the present value of the strike and dividend (i.e. $(X + d_1)B(0, t_1) = \$15.382$). The cash inflow upon initiation is $(3.88 - 3.50) = \$0.38$, and a nonnegative terminal cash flow is guaranteed.

Optimal Exercise Policy For Put Options

Unlike call options, it may be optimal to exercise put options prior to expiration. To see this, consider a stock whose value falls to zero. In this case the put holder should exercise the option immediately. This follows, since if the investor delays action, the interest received from the strike price is lost. This leads to property 8

Property 8

Early exercise of in the money put options may be optimal.

Immediately after an ex-dividend date, the holder of an in-the-money put option who also owns the stock may exercise the option. This is especially likely if the put is deep in the money and no more dividends are to be paid prior to expiration. By not exercising the option, the holder foregoes the interest that could be obtained from investing the strike price. Just prior to an ex-dividend date, early exercise of a put option is never optimal. Since dividends cause the stock price to drop further, investors will always prefer delaying exercising to just after the ex- dividend date.

Since early exercise of American put options is a real possibility, these contracts will be more valuable than their European counterparts. Moreover, because American put options can be more valuable exercised than not exercised, it must follow that on occasion European put options could have values less than their intrinsic values. In other words, it is possible for European put options to command a negative time premium. This fact will be reconsidered later.

Property 9

Immediate exercise of a put on a dividend paying stock is not optimal if the size of the dividend exceeds the interest income on the strike from the current date to the ex-dividend date.

Proof: Recall that in the absence of dividend payments, early exercise of a put may be optimal because interest income can be earned on the proceeds obtained as soon as the option is exercised. Deferring exercise implies interest income is being foregone. For a dividend paying stock, delaying exercise also results in foregoing interest income. However, by exercising early the investor does not profit from the discrete downward jump in the stock price that occurs at the ex-dividend date. If the interest earned on the strike from now until the ex-dividend date is smaller than the size of the dividend, then clearly it is beneficial to wait.

For a dividend paying stock, we can obtain a bound on the put price by considering the strategy of delaying exercise until just after the ex-dividend date. Specifically, viewed from time $t, t < t_1$, we have, from Property 7:

$$P(t) \geq (X + d_1)B(t, t_1) - S(t_1)$$

Clearly immediate exercise at time t is not optimal if

$$X - S(t) \leq (X + d_1)B(t, t_1) - S(t_1)$$

which upon simplification reduces to

$$\begin{aligned} d_1 B(t, t_1) &\geq X[1 - B(t, t_1)] \\ d_1 &\geq X[e^{r(t_1 - t)} - 1] \end{aligned}$$

The right hand side of this equation represents the interest income generated by the strike price from date t to t_1. Note that as t tends to t_1, the right hand side of this equation tends to zero and the condition will eventually be satisfied. Let t^* be chosen such that

$$d_1 = X[e^{r(t_1 - t^*)} - 1].$$

Then, over the interval $[t^*, t_1]$ early exercise of an American put will never be optimal.

If the current time t, falls in the interval $[t^*, t_1]$, then early exercise, prior to t_1, is not optimal. If the current time t falls outside this interval i.e. if $t < t^*$, then early exercise may be optimal. Indeed if the stock price is sufficiently low, or if interest rates are sufficiently high, then early exercise is a distinct possibility.

The above result generalizes to the case where multiple dividends occur prior to expiration. In this case the interval $[t^*, t_1]$ will be wider than the interval obtained by ignoring all but the first ex-dividend date, because the effect of the additional dividends provides increased incentives to delaying exercise.

A direct implication of this property is that the probability of exercise decreases as the size of the dividend increases.

Strike Price Relationships For Call and Put Options

The next property provides the relationships between options with different strike prices.

Property 10

Let C_1, C_2, C_3 (P_1, P_2, P_3) represent the cost of three call (put) options that are identical in all aspects except strike prices. Let $X_1 \leq X_2 \leq X_3$ be the three strike prices, and for simplicity, let $X_3 - X_2 = X_2 - X_1$. Then, to prevent riskless arbitrage strategies from being established, the option prices must satisfy the following conditions:

$$\begin{aligned} C_2 &\leq (C_1 + C_3)/2 \\ P_2 &\leq (P_1 + P_3)/2 \end{aligned}$$

Proof: Consider portfolio A, containing two call options with strike X_2, and portfolio B, containing one call option with strike $X_1$and one call option with strike X_3. Assuming the portfolios are held to expiration, Exhibit 5 illustrates the profits.

Exhibit 5

Strike Price Relationships for Call Options

Portfolio Value	$S_T \leq X_1$	$X_1 < S_T \leq X_2$	$X_2 < S_T \leq X_3$	$S_T > X_3$
A: $2C_0(X_2)$	0	0	$2(S_T - X_2)$	$2(S_T - X_2)$
B: $C_0(X_1) + C_0(X_3)$	0	$(S_T - X_1)$	$(S_T - X_1)$	$(S_T - X_1)$ $+(S_T - X_3)$
		$V_A(T) \leq V_B(T)$	$V_A(T) \leq V_B(T)$	$V_A(T) = V_B(T)$

Since the future value of portfolio B is always at least as valuable as portfolio A, it must follow that the current value of $B, V_B(O)$, is no less than that of $A, V_A(O)$. That is, $V_B(O) \geq V_A(O)$. Hence $C_1 + C_3 \geq 2C_2$, from which the result follows.

If $C_2 > (C_1 + C_3)/2$, then an investor would sell portfolio A and buy portfolio B. The initial amount of money received would be $2C_2 - C_1 - C_3$. If held to expiration, additional arbitrage profits might become available. If, however, the X_2 options were exercised prior to expiration, when the stock was at S^*, the amount owed could be obtained by exercising both the X_1, and X_3 calls. Hence, regardless of what price occurs in the future, riskless arbitrage strategies become available unless the middle strike call is valued less than the average of its neighboring strikes. A similar result holds for put options.

Example

Consider the following option contracts:

Strike	Price
40	4
45	C_0
50	0.88

The upper bound for the 45 call option contract should be $C_0 \leq \frac{4+0.88}{2} = \$2.44$.

Property 10 can be generalized. Specifically option prices are convex in the exercise price. That is, if $X_1 < X_2 < X_3$, then

$$C(X_2) \leq \lambda C(X_1) + (1 - \lambda)C(X_3)$$

$$P(X_2) \leq \lambda P(X_1) + (1-\lambda)P(X_3)$$

where $\lambda = (X_3 - X_2)/(X_3 - X_1)$.

Put-Call Parity Relationships

In this section we shall develop pricing relationships between put and call options.

European Put-Call Parity: No Dividends

Property 11

With no dividends prior to expiration, a European put should be priced as a European call plus the present value of the strike price less the stock price. That is:

$$P_0^E = C_0^E + XB(0,T) - S_0$$

where the superscript E emphasizes the fact that the options are European.

Proof: Under the assumption of no dividends and no premature exercising consider portfolios A and B, where A consists of the stock and European put and B consists of the European call with X pure discount bonds of face value $1 that mature at the expiration date. Exhibit 6 compares their future values.

Exhibit 6 : Put-Call Parity

Portfolio	Current Value	Value at date T	
		$S_T < X$	$S_T \geq X$
A	$P_0 + S_0$	X	S_T
B	$C_0 + XB(0,T)$	X	S_T
		$V_A(T) = V_B(T)$	$V_A(T) = V_B(T)$

To avoid riskless arbitrage opportunities, it must follow that the current value of the two portfolios are the same. This leads to the result. This relationship is called the *European put-call parity equation.*

Example

Given the price of a stock is $55, the price of a three-month 60 call is $2, and the riskless rate is 12 percent, then the lower bound for a three- month 60 put option can be derived.

Specifically,

$$\begin{aligned} P_0 &= C_0 + XB(0,T) - S_0 \\ &= 2 + 60e^{-0.12\times(3/12)} - 55 \\ &= \$5.23 \end{aligned}$$

European Put-Call Parity Equations with Dividends

Property 12

Consider a stock that pays a single dividend of size d_1 at time $t_1 < T$. An European put option is priced as follows:

$$P_0^E = C_0^E + XB(0,T) + d_1 B(0,t_1) - S_0$$

Proof: Consider a stock that pays a dividend d_1, at time t_1. Consider the following two portfolios, A and B. Portfolio A contains the stock and put option. Portfolio B contains the call and X pure discount bonds of face value \$1 that mature at the expiration date, along with d_1, pure discount bonds that mature at time t_1. Exhibit 7 illustrates the terminal values under the assumption that all dividends or payouts are reinvested at the riskless rate.

Exhibit 7

Arbitrage Portfolio-Put-Call Parity with Dividends

Portfolio	Current Value	Value at date T	
		$S_T < X$	$S_T \geq X$
A	$P_0 + S_0$	$X + d_1 G(t_1,T)$	$S_T + d_1 G(t_1,T)$
B	$C_0 + XB(0,T) + d_1 B(0,t_1)$	$X + d_1 G(t_1,T)$	$S_T + d_1 G(t_1,T)$
		$V_A(T) = V_B(T)$	$V_A(T) = V_B(T)$

Since the terminal values of the two portfolios are the same regardless of the future stock price, it must follow that, to avoid riskless arbitrage opportunities, the current portfolio values must be the same. This then leads to the result.

The put-call parity equation not only values a European put option in terms of stocks, calls, and bonds, but also provides a mechanism for replicating a put option. That is, a portfolio containing one call option, X pure discount bonds maturing at time T, d_1 pure discount bonds maturing at time t_1, and a short position in the stock completely replicates the payouts of a put option.

Indirect Purchasing of Puts: A Synthetic Put

From put-call parity:

$$P_0 = C_0 + XB(0,T) - S_0.$$

Hence a portfolio consisting of a European call option together with X pure discount bonds and a short position in the stock produces the same payout as an European put. This portfolio is called a *synthetic* put.

Indirect Purchasing of Stock: A Synthetic Stock

From put-call parity:

$$S_0 = XB(0,T) + C_0 - P_0.$$

Hence, rather than buy the stock directly, one can buy it "indirectly" by purchasing X discount bonds maturing at time T, by buying a call option, and by selling a European put.

Indirect Purchasing of Calls: A Synthetic Call

Since $P_0 = C_0 + XB(0,T) - S_0$, we have

$$C_0 = S_0 + P_0 - XB(0,T)$$

Thus, a call option can be replicated by buying the stock and put option and borrowing the present value of the strike.

American Put-Call Parity Relationships: No Dividends

Property 13

With no dividends prior to expiration, the value of an American put option is related to the American call price by

$$P_0 \geq C_0 + XB(0,T) - S_0$$

Proof: Property 13 follows directly from Property 9 and the recognition that American puts are more valuable than European options (that is, $P_0 \geq P_0^E$). Note that since $C_0^E = C_0 \geq 0$, we have $P_0^E \geq C_0 + XB(0,T) - S_0$. That is, a European put should always be priced no lower than the present value of the strike price less the stock price. Note that this lower bound is less than the intrinsic value, $X - S_0$. European put options could have negative time premiums.

Property 14

With no dividends prior to expiration, the value of an American put option is restricted by:

$$P_0 \leq C_0 - S_0 + X$$

Proof: Consider portfolio A, which contains a call and X dollars invested in the riskless asset, and portfolio B, which contains a put option together with the stock. Exhibit 8 shows the cash flows of the two portfolios. Since the value of portfolio B depends on whether the put was exercised prematurely, both cases must be considered.

Exhibit 8
Bounding American Put Option Prices

Portfolio Value	Value at date t (if the put is exercised)	Value at date T	
		$S_T < X$	$S_T \geq X$
A: $C_0 + X$	$C_t + XG(0,t)$	$XG(0,T)$	$(S_T - X) + XG(0,T)$
B: $P_0 + S_0$	X	X	S_T
	$V_A(T) > V_B(T)$	$V_A(T) > V_B(T)$	$V_A(T) > V_B(T)$

If the put is exercised early, the stock is delivered in receipt for \$X. Note that, in this case, the value of the bonds alone in portfolio A exceeds the value of portfolio B. If the put is held to expiration, then portfolio A will be more valuable than portfolio B, regardless of the future stock price. Clearly, to avoid risk-free arbitrage opportunities, the current value of portfolio A must be no smaller than the value of portfolio B. Hence, $C_0 + X \geq P_0 + S_0$, from which the result follows.

Example

The stock trades at \$60, a three month at the money call trades at \$2, and the riskless rate is 12 percent. Then, the at the money put is constrained by: $P_0^A \leq C_0 + X - S_0 = 2 + 60 - 55 = 7$, and $P_0^A \geq C_0 + XB(0,T) - S_0 = 5.23$. Hence, $5.23 \leq P_0^A \leq 7$.

Other Pricing Relationship Properties

In this section we state several additional properties. The proofs are left as exercises.

Property 15

(i) The difference between the prices of two otherwise identical European calls (puts) cannot exceed the present value of the difference between their strike prices.

$$\begin{aligned} C^E(X_1) - C^E(X_2) &\leq (X_2 - X_1)B(0,T) \\ P^E(X_2) - P^E(X_1) &\leq (X_2 - X_1)B(0,T) \end{aligned}$$

(ii) The difference between the prices of two otherwise identical American calls (puts) cannot exceed the difference between their strike prices.

$$\begin{aligned} C(X_1) - C(X_2) &\leq (X_2 - X_1) \\ P(X_2) - P(X_1) &\leq (X_2 - X_1) \end{aligned}$$

where $X_2 \geq X_1$.

Property 16

(i) The difference in prices between two American calls and two American puts is bounded below as follows:

$$[C(X_1) - C(X_2)] - [P(X_1) - P(X_2)] \geq (X_2 - X_1)B(0,T)$$

where $X_2 \geq X_1$. This relationship is called the box spread lower boundary condition.

Property 17

Option prices are homogeneous of degree one in the stock price and strike price. That is:

$$\begin{aligned} C(\lambda S, \lambda X, T) &= \lambda C(S, X, T) \\ P(\lambda S, \lambda X, T) &= \lambda P(S, X, T) \end{aligned}$$

The property is rather obvious for the case when the numeraire is just rescaled. For example, if the stock is measured in cents rather than dollars, then, as long as the strike is measured in cents as well, the option price remains correctly priced in the new numeraire of cents. Property 17 implies that when a stock splits, the strike price can be adjusted without creating problems.

Example

Assume a stock is priced at \$110. Acall trades with strike \$100. If the stock splits into two shares of \$55 each, then the terms of the option contract has to be modified. From this property, an appropriate adjustment would be to exchange the original call option with two options each controlling one new share at a strike price of \$50 per share. The value of these two options would equal the value of the original single contract.

One has to be careful in applying this property in certain cases. For example, if the stock returns process is not independent of the level of the stock price, then the property is not valid. As an extreme example, consider a firm that had stock priced at \$100 and an option with strike \$120. Now assume the firm changes its strategy from a risky to a riskless one in such a way that the new stock price, is \$50. Say the new strike is set at \$60. With riskless rates equal to 5% per year, there is no chance for the option to expire in the money, and the call option will be worthless. However, if the returns process is independent of the stock price process then the above property holds.

Property 18

A portfolio of options is never worth less than an option on a portfolio. Specifically:

$$\begin{aligned} C(\sum_{i=1}^{n}\lambda_i S_i, X, T) &\leq \sum_{i=1}^{n}\lambda_i C(S_i, X, T) \\ P(\sum_{i=1}^{n}\lambda_i S_i, X, T) &\leq \sum_{i=1}^{n}\lambda_i P(S_i, X, T) \end{aligned}$$

This property states that a put option that provides protection against adverse moves in the portfolio value is less costly than a portfolio of put options that insures each component stock against loss. Of course, the terminal payouts of these two strategies are different, with the portfolio of options providing higher payouts in certain states. These higher payouts are reflected in the higher cost of the strategy.

Conclusion

In this chapter arbitrage arguments have been used to derive bounds on option prices. In addition, the relationship between put and call options has been explored and optimal call and put exercise strategies have been investigated. Other bounds can be derived. The important point however, is that the set of option prices are constrained to move together in certain ways. As soon as their relative prices deviate, riskless arbitrage possibilities exist.

Several empirical studies have been conducted to test the results in this chapter. Some of this empirical research will be discussed in future chapters. However, the results have shown that the bounds do hold. More precisely, only rarely are the relationships violated, and when they do, profits from implementing the appropriate strategies are insignificant, especially when trading costs are considered.

References

This chapter draws very heavily on Merton's article, published in 1973. The beauty of these arbitrage arguments stems from the fact that they require only that investors prefer more wealth to less. If additional assumptions are placed on investor preferences, tighter bounds on option prices can be derived. Examples of such approaches include Parrakis and Ryan; Ritchken; Ritchken and Kuo; and Levy. Jarrow and Rudd provide a comprehensive treatment of bounds for cases when dividends are uncertain and interest rates are random. Cox and Rubinstein also provide a rigorous treatment of this subject. Empirical tests of these bounds are discussed by Gould and Galai; Klemkosky and Resnick; Stoll; and the survey by Galai.

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Exercises

1. A and B are both priced at 50. In the up-state A is 65, B is 60. In the down-state A is 40 and B is 35.

a. Construct a riskless arbitrage portfolio.

b. In an efficient market, what would happen to the prices of A and B?

2. XYZ trades at \$50. One-dollar, face-value bonds currently trade at \$0.90. Assuming the bonds mature at the expiration date of the option, compute the lower bound price of an at-the-money call option.

3. XYZ is currently trading at \$50. It is about to declare a\$0.50 dividend. An at the-money, three-month call option is trading at \$0.70. The riskless rate is 10 percent.

a. Compute the lower bound on the option price.

b. Construct a strategy that yields riskless arbitrage opportunities.

c. How large must the dividend be before the investor would have to consider the possibility of early exercise?

4. Explain why call options will be exercised (if at all) just prior to an ex- dividend date, while put options will not be exercised at this time.

5. XYZ pays no dividends and trades at \$50. The riskless rate r is 10 percent. What is the lowest price at which a six-month put with strike price 55 should be sold? If the put traded below this value, what could be done to lock into a guaranteed profit?

6. The six-month call prices on XYZ, which currently trades at \$50, are shown below. XYZ pays no dividends.

Strike	Price
45	5.2
50	4.5
55	3.0
60	0.50

Construct a portfolio that will produce riskless profits.

7. The call price on a six-month XYZ 50 call option is \$6 and the riskless rate is 10 percent.

a. Compute the lower bound on a six-month 50 put option. Assume the stock price is \$52.

b. If the underlying stock declares a dividend of \$1 two months prior to expiration, would the lower bound increase, decrease, or remain the same? Explain.

8. XYZ trades at \$60. The market price of an XYZ January 50 put is 1/16. The market price of the January 50 call is \$11. Consider the strategy of buying the stock and put option and selling the call. Compute the initial investment and explain why a profit equal to the strike price less the initial investment is guaranteed.

9. Given XYZ trades at \$30, the riskless rate of return is 12 percent, and an at-the-money call option with six months to expiration is trading at \$4.

a. Compute the price of a European put option.

b. By plotting a profit function, show that the payouts of a stock can be replicated by a portfolio containing the six-month at-the-money call, a short position in the put, and borrowing funds at the riskless rate. How much needs to be borrowed?

10. Using arbitrage arguments alone, show that an American call option cannot be valued more than a stock.

11. Explain why a European put option could have a negative time premium, while an American put option could not.

12. Prove Property 15.