Pricing Options on
Dividend paying stocks, FOREX, Futures, Consumption Commodities

- The Black-Scholes Model
- The Binomial Model and Pricing American Options
- Pricing European Options on dividend paying stocks
- Pricing European Options on Stock Indices
- Pricing European Options on FOREX
- Pricing European Options on Futures
- Pricing European Options on consumption commodities
- Pricing American Options on the above.
- Pricing simple real options
European Call Options on a Stock

The Black Scholes Equation

\[ C_0 = H^* S_0 - B^* \]

where

\[ H^* = N(d_1) \]
\[ B^* = X e^{-rT} N(d_2) \]

and

\[ d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma \sqrt{T}} \]
\[ d_2 = d_1 - \sigma \sqrt{T}. \]

\( N(x) \) is the probability that a standard normal random variable (usually represented by \( Z \)) is less than \( x \).

Example: Pricing a European Call Option Using The Black Scholes Model

- Consider the theoretical price of a 3 month call option on a stock priced at $50. The strike price is 45, the riskless rate is 6% and the volatility is 20% per year. Hence, 
  \( S(0) = 50, X = 45, T = 0.25, \sigma = 0.20, \) and \( r = 6\% \) Then, the Black Scholes call price can be computed as follows.

\[ d_1 = \frac{\ln(50/45) + (0.06 + 0.20^2/2)\times 0.25}{0.20 \times \sqrt{0.25}} = 0.05 \]
\[ d_2 = d_1 - 0.20 \sqrt{0.25} = 0.426 \]

Then, using standard normal tables, 
\( N(d_1) = 0.742, N(d_2) = 0.665, \) and we obtain

\[ H^* = N(d_1) = 0.742 \]
\[ B^* = 45 e^{-0.06 \times 0.25} N(d_2) = 45 e^{-0.06 \times 0.25} (0.665) = 29.48 \]
\[ C(0) = H^* S(0) - B^* = 50(0.742) - 29.48 = 87.62 \]
What do you need to price an option

- Underlying S(0)
- Strike, X
- Time to expiration, T
- Riskless rate, r
- Volatility

Estimating volatility
- Use Historical data
- Use Implied Volatility

Option Models

- Give us Theoretical Prices, when none exist.
- Give us prices which we can use to compare with market prices.
- Allows us to establish strategies. (Buy underpriced/sell overpriced)
- Gives us the replicating portfolio.
- Allows us to imply out volatility information.
- Test of models and market efficiency.
- Models can be used to price corporate securities and many real assets
Historical Volatility vs Implied Volatility

- What is implied volatility?
- Is it a good predictor of actual future volatility?
- Which Implied Volatility?
- Tests of Option Models
- The Volatility Smile.

Example: Pricing a Call Option Using a 4 - Period Binomial Approximation

Consider a 1 year at-the-money American call option on a non-dividend paying stock. The stock price is $100, the risk free rate is 10%, and the annual volatility is 39.72% per year. Using four partitions, \( n = 4, T = 1.0 \), yields \( \Delta t = 0.250 \) and

\[
\begin{align*}
    u &= e^{\sigma \sqrt{\Delta t}} = 1.2197 \\
    d &= e^{-\sigma \sqrt{\Delta t}} = 0.8199 \\
    \theta &= \frac{e^{\sigma \Delta t} - d}{u - d} = 0.51379
\end{align*}
\]
Example: Pricing a Put Option Using a 4-Period Binomial Approximation

- Consider a 1 year at-the-money European put option on the previous stock. Following the backward recursion from the terminal period yields the put prices indicated below the stock price.

- Notice that the option price can fall below the intrinsic value, since early exercise is not permitted.

- Specifically, when the option is deep in-the-money, early exercise may be advantageous since the strike price can be obtained early and can be used to generate interest income.
Example: Pricing an American Put Option

Figure 17: Stock and American Put Option Prices
To price an option in the Binomial lattice we replace the growth rate of the stock by the riskless rate.

The lattice converges to a Geometric Wiener process.

For European options we can use simulation, where we replace the growth rate by the riskfree rate.

Simulation, the Binomial lattice and the Black Scholes model, give the same results for pricing an European option.

The Binomial lattice is good for American options.

Simulation is good for exotics.

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Options on Stocks paying dividends

- Assume a stock pays out $d$ at a time before the expiration date,
- $S(0) = PV(d) + PV(\text{all other future expected dividends})$
- The call holder has a claim on the second component.
- $S(0) = PV(d) + G(0)$
- After the dividend at date $t$ say, $S(t) = G(t)$
- The call holder has a claim on $G(T)$. The current value of $G(T)$ is $G(0)$.
- Use Black Scholes with $G(0)$ not $S(0)$. 

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Options on Stocks paying Continuous dividends.

- Total expected return = dividend yield + price appreciation
- Need to reduce the growth rate by the dividend yield.
- So for a simulation, we have:

\[
S(t) = S(0) e^{\alpha T + \sigma \sqrt{T} Z_T}
\]

\[
\alpha = r - q - \sigma^2 / 2
\]

European options on stocks paying continuous dividends

- A stock valued at \( S(0) \) paying dividends continuously has the same value at date \( T \) as a stock paying no dividends would have if its date 0 value was \( G(0) = S(0) \exp(-qT) \).

- To price an European option use \( G(0) \) instead of \( S(0) \).
Example

- The dividend yield on the S&P 500 over the next month is estimated as 4.2% per year. The index is at 300. The following data is given:
  - Strike 300
  - $R = 6\%$
  - $T = 1$ month

- $G(0) = 300 \exp(-0.042/12) = 298.95$
- $C(0) = 10.47$.

European Options on FOREX

- $\mu = r - q$
- $\alpha = \mu - \sigma^2/2$
- $q = r_f$
- $G(0) = S(0)e^{-\eta T}$

- Use $G(0)$ instead of $S(0)$ in the BS equation.
European options on Futures
(The Black Model)

- What should the expected price appreciation on a Futures contract be?
- Hint: The initial investment is zero?
- Yes! The growth rate should be zero!

$$\mu = r - q$$
$$\alpha = \mu - \sigma^2 / 2$$

So take $q = r$
$G(0) = S(0) \exp(-rT)$

European Calls on Consumption Commodities

- A commodity pays a convenience yield
- The convenience yield is a non observable dividend
- Can be extracted from the forward price.
- Let $u$ be the annualized convenience yield over the time period to the expiration date.
- Use $G(0) = S(0)\exp(-uT)$ in the Black Scholes Equation.
Pricing American Options on a lattice

- Compute \( u \) and \( d \) the same way.
- Use

\[
p = \frac{e^{(r-q)\Delta t} - d}{(u - d)}
\]

Watch out for early exercise.

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Pricing a Real Option

- You have the option to buy a building for 1m dollars. The option expires in one year.
- The building provides a rental income of 5%
- The riskless rate is 8%
- What is the value of the option.
Pricing a Real Option

- You have the option to buy a building for 1m dollars. The option expires in one year.
- The building provides a rental income of 5%
- The riskless rate is 8%
- What is the value of the option.
- The current purchase price is \( S(0) = 1 \)m dollars.

- The growth rate of the building in a risk neutral setting is (8-5)%
- If the option is European, set \( G(0) = S(0) \exp(-0.05) \).
- Use Black Scholes!

Conclusion

- How to Use Black Scholes
- Black Scholes and the Binomial lattice
- Dividends - discrete
- Dividends- continuous
- Pricing options on Indices, FOREX, Futures, Consumption commodities, real assets.
- American options and early exercise.
- Simulation, Lattices, and Black Scholes.