9

Measures of Price Sensitivity 1

This chapter reviews the factors that cause bond prices to be volatile. The Macaulay measure of duration and modified duration are described. This latter measure captures the exposure of a bond to interest rate rmoves of a certain kind. Immunization strategies based on duration matching and duration-convexity matching are presented. The limitations of these approaches are discussed. Measures of risk due to a twisting term structure are investigated. The basic measures of duration, convexity and twist risk are helpful in characterizing risk exposures.

For most of this chapter we assume the initial yield curve is flat. That is, the yields to maturity are the same for all maturities. We also assume that when unanticipated information arrives causing the yield curve to change, the change is the same for all maturities. Hence yield curves remain flat, and just move up and down, depending on information. While this assumption is very unrealistic it provides a start for our analysis. Later on in this chapter we will allow the initial yield curve to be arbitrary, but we will assume that all shocks to the yield curve are the same. In this case the yield curve never changes its basic shape, although it again moves up and down as information arrives. If non parallel shocks, such as twists occur, then our measures of risk need to be reassessed. In future chapters we will consider alternative risk measures that can handle alternative types of shocks to the yield curve.

The purpose of this chapter is

- To describe measures of duration and convexity in regard to bond price volatility,
- To discuss the use of duration and convexity measures in imunization strategies,
- To discuss other measures of interest rate sensitivity, including the dollar value of a basis point shock, and
- To provide the first step in establishing a framework for interest rate risk management.

9.1 PRICE-YIELD RELATIONSHIPS

Changes in the yield curve tend to affect the price of some fixed-income securities more than others. The sensitivity of bond prices to interest rate change depends on many factors, including current yields and yield chages, time to maturity, and coupon size.

Effect of Yield Change

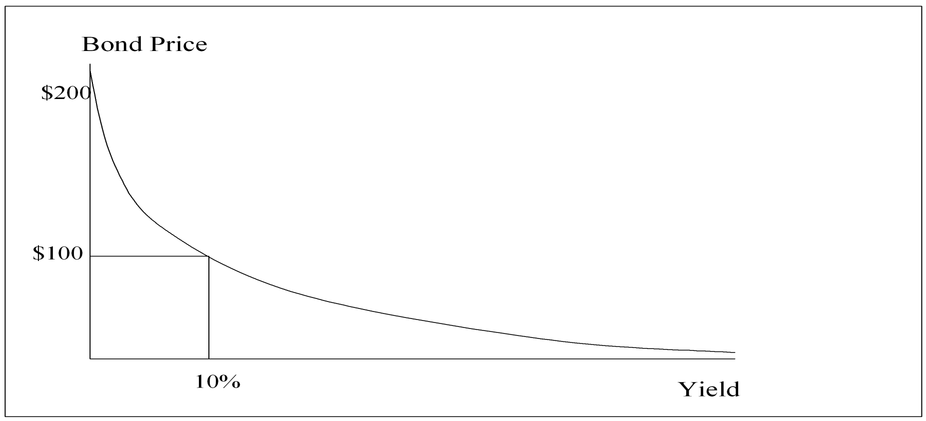
Figure 9.1 shows the typical relationship between the price of a coupon bond and the yield to maturity.

Assume the coupon is 10% per year paid semiannually, and that the bond has ten years to maturity. If the yield on the bond was 10%, then the bond would be priced at its par value of \$100. If yields were zero, then there is no time value for money, and the value of the bond would equal the value of all the cash flows, namely, $100 + 5 \times 20 = \$200$. Finally, as the yield goes to infinity, the value of the bond goes to zero. We see, then, that the price of a bond is convex in the yield. This means that the sensitivity of a bond to changes in the yield, will depend on the actual level of rates. A one basis point change in yields, when the yield is low has a much bigger impact on the price, then when the yield is low.

Due to the convex relationship between prices and yields, for a large decrease in yield, the percentage increase in price is greater than the percentage decrease in price for an equal increase in yield. That is, prices increase at an increasing rate as yields fall, and decrease at a decreasing rate when rates rise.

Fig. 9.1 Price vs Yield on a Coupon Bond

Exhibit 1:
Bond Price versus Yield



Example

Consider a four-year 8% bond with annual coupons sold at par ($1000) to yield 8%. If yields fall to 6%, the bond price is

$$B = \frac{80}{1.06} + \frac{80}{1.06^2} + \frac{80}{1.06^3} + \frac{1080}{1.06^4} = \$1069.30.$$

This yield change causes a 6.93% change in the bond price. If interest rates rise to 10%, the bond price is

$$B = \frac{80}{1.10} + \frac{80}{1.10^2} + \frac{80}{1.10^3} + \frac{1080}{1.10^4} = \$936.60.$$

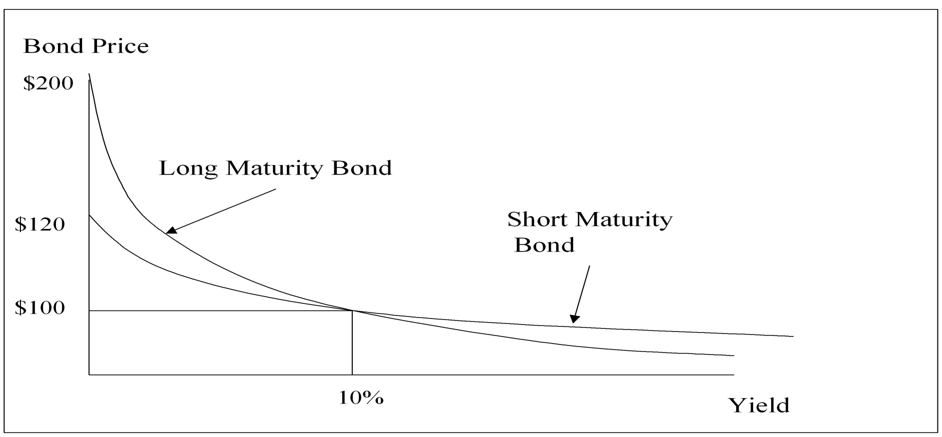
This yield change causes a 6.34% change in bond price. Thus, a decrease in yields causes a larger percentage change in the price than an equivalent increase in yields.

Effect of Maturity on Bond Prices

Figure 9.2 shows the price yield relationship of two bonds that have the same coupons and yield, but different maturities. The coupons are 10%. If the yield were 10%, both bonds would be priced at par. If yields were zero, then the short term bond with two years to maturity, would be worth $100+5\times4 = \$120$, while the 10 year bond would be worth $100 + 5 \times 20 = \$200$. As the curve shows, it appears that the longer term bond is more sensitive to yield changes.

Fig. 9.2 Effect of Maturity on Bond Prices

Exhibit 2:
Sensitivity of Prices of Bonds with Different Maturities to Yield



Indeed, in most cases a given change in yield will cause a longer term bond to change more in percentage terms than a shorter-term bond. For some discount bonds, however, the percentage change in prices for a given decrease in yield to maturity increases with maturity up to a point and then decreases with maturity once maturity is large enough.

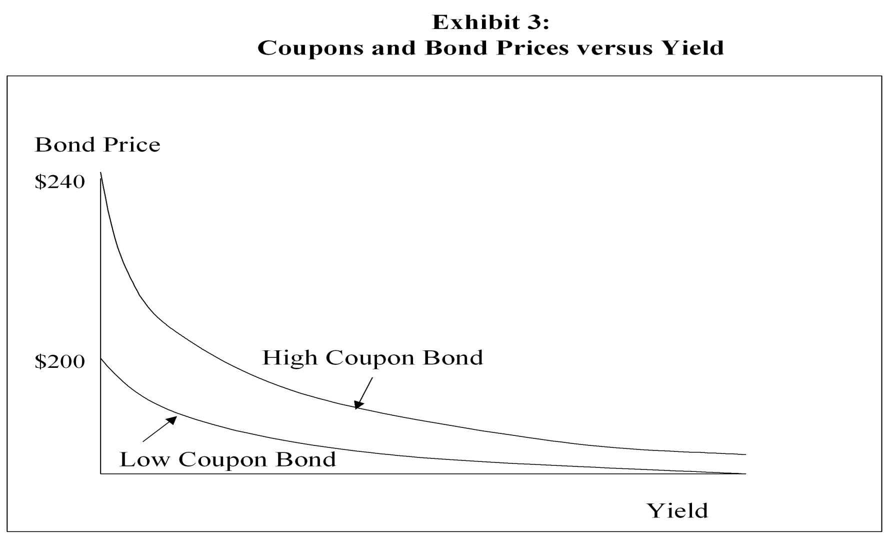
Example

Consider two 5% coupon bonds, both priced to yield 8%. One is a four-year bond, the second an eight-year bond. Both bonds pay interest annually. The shorter-term bond is priced at \$900.63, while the longer-term bond is priced at \$827.60. Assume yields rise to 10%. Then, from the bond pricing equation, the four-year bond will be priced at \$841.50, while the eight-year bond will be priced at \$733.40. In percentage terms, the decline in price of the shorter-term bond is 6.6%, compared to 11.4% for the longer-term bond.

Effect of Coupon Size on Bond Prices

Figure 9.3 compares the the price yield relationship for a 10 year bond that has coupons of 14% per year with that of an otherwise identical bond with a coupon of 10%. If the yield was 14%, then the first bond would be priced at par. Moreover, if the yield were zero, then the price would be $100 + 7 \times 20 =$ \$240.

Fig. 9.3 Effect of Coupons on Price



Since this bond has all the features of the lower coupon bond, except that it pays out more, it has to be the case that its price yield curve lies above the curve of the lower coupon bond. Which of the above two bonds has the greater risk? Put another way, given a change in yields, which bond will change more in percentage terms?

A given change in yields will cause the price of the lower-coupon bond to change more in percentage terms. The reason for this follows from the fact that higher-coupon bonds, having greater cash flows, return a higher proportion of value earlier than lower-coupon bonds. This implies that relatively less of the high-coupon bond faces the higher compounding associated with the new discount factor. Therefore, on a relative basis, less price adjustment is required for the higher-coupon bond.

Example

Consider two four-year annual coupon bonds, both priced to yield 8%. The first bond has a 5% coupon, the second a 10%. From the bond pricing equation, their prices are \$900.63 and \$1066.24, respectively. Assume interest rates change so that each bond is now priced to yield 10%. Then the new bond prices are

$$
\begin{aligned}
B_1 &= \frac{50}{1.10} + \frac{50}{1.10^2} + \frac{50}{1.10^3} + \frac{1050}{1.10^4} = \$841.50 \\
B_2 &= \frac{100}{1.10} + \frac{100}{1.10^2} + \frac{100}{1.10^3} + \frac{1100}{1.10^4} = \$1000
\end{aligned}
$$

In percentage terms, the 5% coupon bond has changed by $(900.63-841.50)/900.63) =$ 6.6% while the 10% coupon bond has changed by 6.2%. In general, low-coupon bonds are more sensitive to yield changes than high-coupon bonds.

Bonds trading above their face value (premium bonds) have higher coupon rates than bonds trading below their face value (discount bonds) and, hence, all things being equal, will be less sensitive to yield changes.

In summary, the price sensitivity of a coupon bond is affected by its coupon rate and maturity as well as the current level of yield. In general, for a given maturity, the lower the coupon rate the greater the volatility, and for a fixed coupon, the greater the maturity the greater the volatility. To compare the risk of bonds with different coupons and different maturities a measure called duration is required. This is considered next.

9.2 MACAULAY DURATION

Since high coupon bonds provide a larger proportion of total cash flow earlier in the bond's life than lower coupon bonds with the same maturity, they are effectively shorter term instruments. As a result, the actual maturity date of the bond is not necessarily a good measure of the length of a coupon bond.

To obtain a more meaningful measure it is helpful to first represent the bond as a portfolio of discount bonds and to measure the maturity of each cash flow. From the bond pricing equation:

$$B_0 = \sum_{t=1}^{m} P(0,t)CF_t.$$

Let w_t be the present value contribution of the t^{th} cash flow to the bond price. Then

$$w_t = \frac{P(0,t)CF_t}{B_0} \quad \text{for } t = 1, 2, ..., m$$

The duration, D, of a bond is just the weighted average number of periods for cash flows for this bond. That is

$$D = \sum_{t=1}^{m} t \times w_t$$

Notice that the greater the time until payments are received, the greater the duration. If the bond is a discount bond, all payment is deferred to maturity and the duration equals maturity. For bonds making periodic coupon payments the early payments will reduce the duration away from the maturity.

Macaulay Duration is affected by changes in the market yield, the coupon rate and the time to maturity.

Example

Consider a 4-year bond paying a 9% coupon semi-annually and priced to yield 9%. The cash flows every 6 months are shown below, as well as the weights and duration calculation.

Period t	Cash Flow CF_t	Present Value $P(0,t)CF_t$	Weighted Present Value $tP(0,t)CF_t$
1	45	43.06	43.06
2	45	41.21	82.42
3	45	39.43	118.3
4	45	37.74	150.94
5	45	36.11	180.55
6	45	34.56	207.33
7	45	33.07	231.47
6	1045	734.83	5878.63
Total		1000.0	6892.70

The duration is 6892.70/1000 = 6.89 half years, or 3.45 years.

Example

The sensitivities of durations to changes in yield, coupons, and maturity for a five-year bond that pays 12% annual coupons and yields 12% percent are shown below.

- All factors being equal, the higher the yield, the lower the duration.

Yield	4	8	12	16	20
Duration	4.2	4.1	4.0	3.7	2.9

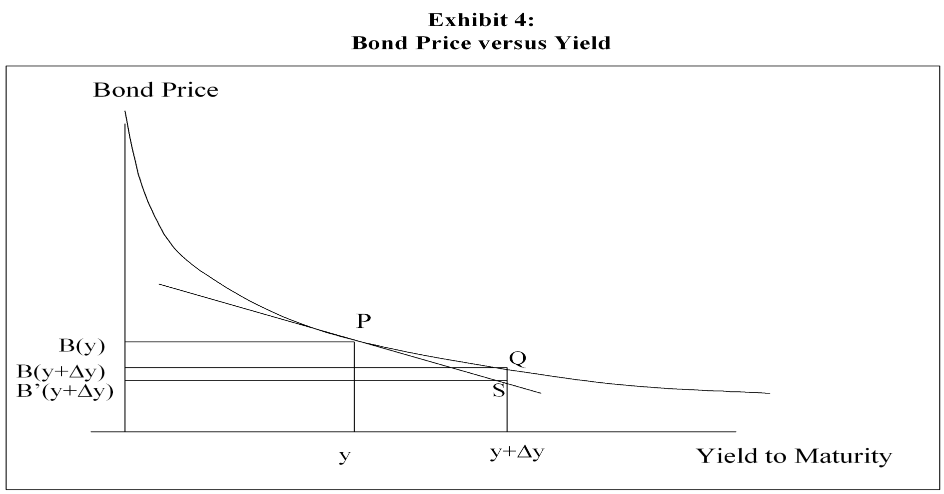
- All factors being equal, the higher the coupon, the lower the duration.

Coupon	4	8	12	16	20
Duration	4.5	4.2	4.0	3.9	3.8

- For this bond the duration increases with maturity. This property is typical, but may not always hold. The exception to this rule will be low coupon bonds with long maturity.

Maturity	3	5	7	10	30
Duration	2.7	4.0	5.1	6.3	9.0

Fig. 9.4 Bond Price versus Yields



9.3 DURATION AND BOND PRICE SENSITIVITY

The price of a coupon bond which pays m coupons, can be written as

$$B = \sum_{t=1}^{m} \frac{CF_t}{(1+y)^t}$$

where y is the current yield-to-maturity of the bond per period and CF_t is the cash flow at date t. Figure 9.4 shows the relationship between the bond price and yield.

The bond price is located at the point P. The slope of the curve at P, dB/dy can be shown to be:

$$\frac{dB}{dy} = -D\frac{B}{(1+y)}$$

If we define modified duration as:

$$D_m = \frac{D}{(1+y)}$$

then

$$\frac{dB}{dy} = -D_m B$$

or

$$\frac{dB}{B} = -D_m dy$$

That is, the instantaneous percentage change in bond price equals the negative of modified duration times the change in yield, provided the change in yield is very small.

If a bond has a modified duration of 4.0 then its market value will change by 4% when the yield per period changes by 1% or equivalently by 100 basis points. The higher the modified duration the more exposed the bond is to interest rate changes.

Example

Consider the previous 4-year bond paying a 9% coupon semi-annually and priced to yield 9% per year, or 4.5% per six-month period. The duration of the bond is 6.89 periods. The modified duration is therefore

$$D_m = D/(1+y) = 6.89/1.045 = 6.593 \text{ periods or } 3.297 \text{ years.}$$

For a one percentage change in the annual yield to maturity, the percentage change in the bond price is 3.297%.

9.4 DURATION OF A BOND WITH UNEVEN PAYMENTS

In our analysis we have assumed that the compounding interval equals the time between successive cash flows. In particular, y is the rate per period. In most cases, coupon payments are equally spaced, except for the first coupon payment. Let p be the fraction of a period till the first coupon date. For example if the time between coupon dates is 6 months, and the time to the first coupon date is 2 months, then with each period corresponding to a six month interval, $p = 2/6$. The Macaulay duration is:

$$D = \sum_{t=1}^{m} (p + t - 1) w_t$$

where

$$w_t = \frac{CF_t}{(1+y)^{p+t} B(0)}$$

is the relative contribution to the bond price made by the present value of the t^{th} cash flow. It can be shown that the modified duration is given by

$$D_m = D_m^* - (1 - p)$$

where D_m^* is the modified duration of the same bond computed with p=1.

Example

To be done

9.5 LINEAR AND QUADRATIC APPROXIMATIONS FOR THE CHANGE IN A BOND PRICE

Assume the yield changes from y to $y + \Delta y$, where Δy is small. The bond price in Exhibit 4 then changes from B(y) (point P) to $B(y + \Delta y)$ (point Q). For Δy sufficiently small, $B(y + \Delta y)$ can be approximated by $B'(y + \Delta y)$ where

$$B'(y + \Delta y) \approx B(y) + \frac{dB}{dy}\Delta y$$

which is indicated by point S in the above diagram. Since $dB/dy = -D_m B$, the above equation can be rewritten as

$$B'(y + \Delta y) \approx B(y) - D_m B(y)\Delta y$$

Note that if Δy is "large" positive or negative, then the linear, or first order approximation, $B'(y + \Delta y)$, is always lower than the actual bond price. The error is attributable to the curvilinear or *convex* relationship between bond prices and yields.

To account for this convexity, we could use a second order approximation, which takes into consideration how the slope changes as the yield y changes. For convex relationships, the change in the slope is always increasing. For example, the slope at the point Q is less negative than the slope at point P. A second order approximation of $B(y + \Delta y)$ which takes into account the curvilinear relationship is given by

$$B''(y + \Delta y) \approx B(y) + \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}(\Delta y)^2$$

This can be rewritten as:

$$B''(y + \Delta y) \approx B(y) - D_m B(y)\Delta y + \frac{1}{2}C(y)B(y)(\Delta y)^2$$

where

$$C(y) = \frac{\frac{d^2B}{dy^2}}{B(y)}$$

is defined as the convexity of the bond. In this case, differentiating the bond price equation two times we obtain:[1]

$$C(y) = \sum_{t=1}^{m} \frac{t(t+1)CF_t}{(1+y)^{t+2}B(y)}$$

The convexity of a straight coupon bond is always positive, implying that the slope of the price yield equation is increasing (becoming less negative) as yields increase.

The quadratic approximation differs from the linear approximation by the last convexity term. Since this term is always positive, the quadratic approximation will always provide higher values than the linear approximation.

For small changes in the yield to maturity, the linear approximation provides a good proxy for the change in bond price. That is, modified duration is a useful measure for price volatility. However, when the market perceives interest rate volatility to be high, then the second order approximation is more precise.

Example

Consider a 5 year bond which pays \$80 in coupons semiannually (i.e., \$40 per 6 months), has a face value of \$1000 and is currently priced to yield 8% per year. The duration of this bond is 8.435 periods (half years), and the modified duration is 8.435/1.04 = 8.111 periods. The convexity of the bond is 80.75 periods squared.

In this example, each period corresponds to a six month interval. To convert the convexity measure to an annual figure requires dividing by the square

[1] In this calculation, the assumption is made that the cash flows are equally spaced, that is, $p = 0$.

of the number of periods in the year. Hence the annualized convexity in this example is $80.75/4 = 20.18$ years squared.

Table 9.1 shows the actual bond price as computed by the bond pricing equation for a variety of changes in the annualized yields to maturity and compares the true prices to the linear and quadratic approximations.

Table 9.1 Linear and Quadratic Approximations of the Bond Pricing Equation

Basis Point Change	Bond Price	Linear Approx.	Error	Quadratic Approx.	Error
-400	1179.65	1162.2	17.45	1179.69	-0.039
-200	1085.3	1081.1	4.20	1085.48	-0.177
0	1000	10000	0	1000	0
200	922.78	918.9	3.88	923.26	-0.477
400	852.79	837.80	14.99	855.248	-2.458
800	731.59	675.60	59.99	745.435	-13.845

The following expressions may be useful for computing bond prices, duration and convexity. The underlying bond pays a coupon of size c for the next n periods.[2] The face value of the bond is F. The yield per period is y. For a semiannual coupon bond, the yield to maturity is therefore 2y.

$$
\begin{aligned}
B(0) &= \frac{c}{y}\left[1 - \frac{1}{(1+y)^n}\right] + \frac{F}{(1+y)^n} \\
D_m &= \frac{c}{y^2 B(0)}\left[1 - \frac{1}{(1+y)^n}\right] + \frac{n(F - c/y)}{B(0)(1+y)^{n+1}} \\
C &= \frac{1}{B(0)}\left[\frac{2c}{y^3}(1 - \frac{1}{(1+y)^n}) - \frac{2cn}{y^2(1+y)^{n+1}} + \frac{n(n+1)(F - c/y)}{(1+y)^{n+2}}\right]
\end{aligned}
$$

The analytical formulae are established using the fact that a coupon bond can be viewed as a combination of an annuity that pays c dollars for n periods, and a discount bond that pays F dollars after n periods. The first term in the bond pricing equation is the value of the annuity. The modified duration equation is then obtained by differentiating this formula with respect to y,

[2] I should generalize this equation with the first fractional period.

and dividing by the bond price. Similarly, the convexity equation is obtained by differentiating this bond pricing equation twice with respect to y, and then dividing by the bond price.

9.6 PRICE VALUE OF A BASIS POINT

A very common risk measure is the sensitivity of the price of a bond to changes in its yield. The price value of a basis point, $PVBP$, sometimes called the dollar value of a basis point, or $DV01$, measures the decline in price associated with a one basis point increase in the yield.

$$DV01 = -\text{Slope of Price Yield Curve } \times 0.01\%$$

or

$$DV01 = -\frac{dB}{dy} \times 0.0001$$

DV01 can be computed directly, by computing the price at the current yield, adding one basis point to the yield, and recomputing the price.

Example

A five year bond pays 10% coupons semi annually and is priced at par. ($y = 0.10$). To compute DV01 we reprice the bond at a yield of 0.1001. This leads to a price of \$99.9614. Hence, $DV01 = 100 - 99.9614 = \$0.0386$.

DV01 can also be computed directly from modified duration. In particular, we have seen that

$$\frac{dB}{B} = -D_m dy$$

Hence

$$dB = -D_m B dy$$

and

$$DV01 = -D_m B \times 0.0001$$

Example

Reconsider our five year bond that pays 10% coupons semi annually and is priced at par. ($y = 0.10$). The modified duration of this bond is $D_m = 3.86$. Hence DV01 $= 100 \times 3.86 \times 0.0001 = \$0.0386$.

9.7 DURATION, CONVEXITY AND DV01 OF A BOND PORTFOLIO

Like the beta value of an equity portfolio, the duration of a bond portfolio, D_p, is computed as the weighted average of the durations of the individual bonds:

$$D_p = \sum_{i=1}^{K} \alpha_i D_i$$

where K is the number of different bonds and α_i is the fraction of portfolio dollars invested in bond i. Similarly, the convexity of a bond portfolio, C_p , is the weighted average of the convexities of the individual bonds.

$$C_p = \sum_{i=1}^{K} \alpha_i C_i$$

The DV01 of a portfolio is defined as the change in value resulting from equal one basis point declines in all yields. Let $DV01_i$ represent the dollar value of a basis point associated with the i^{th} bond. Then

$$DV01_p = \sum_{i=1}^{K} x_i DV01_i$$

where x_i is the number of bonds of type i in the portfolio.

Example

A portfolio consists of 2 bonds. The first position is in three 5 year zero coupon bonds, the second is in one 2 year zero coupon bond. The yield curve is flat at 10%, semiannually compounded. Durations of zero coupon bonds equal their maturities. The modified durations of the two bonds are $\frac{2}{1.05} = xxx$ and $\frac{5}{1.05} = yyy$ respectively. $DVO1_1 = xxx \times 0.0001 \times \frac{100}{(1.05)^10} = qqq$ and $DV01_2 = yyy \times 0.0001 \times \frac{100}{(1.05)^4} = rrr$. Hence

$$DV01_p = 3qqq + 1rrr = \$zzz$$

A one basis point increase will decrease the bond portfolio by $\$zzz$.

9.8 HEDGING UNDER A PARALLEL SHIFT ASSUMPTION

Assume an investor holds a portfolio that has a DVO1 equal to $DV01_p$. Assume that the investor wants to reduce the DV01 to zero. The resulting position will then not change value for a small shock in the yield curve. In order to do this, we assume, at least for the moment that the yield curve shifts up one basis point. That is, there is a parallel shock. Let $DV01_h$ be the DV01 value associated with the hedging instrument. For example, it could be a five year coupon bond. The number of units required of the hedging instrument is chosen such that

$$DV01_p + n_h DV01_h = 0$$

or

$$n_h = -\frac{DV01_p}{DV01_h}$$

The negative sign indicates that a short position in the hedging contracts is necessary.

Example

Assume a position is held that has a DV01 of 500 dollars per million face value. Assume the hedging instrument consists of our 5 year coupon bond yielding 10%. Its DV01 is0.0386 per 100dollars. The number of 5 year bonds to trade is

$$n_h = -\frac{500}{0.0386} = -vvvv$$

By selling short vvv of the five year coupon bonds, the position is immunized from a parallel shock of 1 basis point.

The above example assumes that a parallel shock of one basis point occurs at all maturities. This implies that there is perfect correlation of changes in yields of the two securities. Later on we shall consider the case of hedging when we allow for non parallel shocks in the yield curve.

9.9 IMMUNIZATION OF INTEREST RATE RISK

Consider an investor whose goals require that an investment be dedicated to meet a specific liability of nominal amount F that comes due in m periods. If the investor held a portfolio of discount bonds having face value F and maturity m, then, regardless of interest rate behavior, the future value of the portfolio would cover the future liability. This dedicated portfolio is said to be perfectly immunized since its value is insensitive to changes in the yield curve.

Rather than use a discount bond, assume a coupon bond was held. With coupon bonds, two types of risk exist, *price risk* and *coupon reinvestment risk*. Price risk is the risk that the bond will be sold at a future point in time for a value different from what was expected. Coupon reinvestment risk is the risk associated with reinvesting the coupons at rates different from the yield of the bond when it was purchased. If the maturity date coincides with the holding period, then price risk is eliminated. As the maturity date increases, so does price risk.

As interest rates increase (decrease), bond prices decline (increase) while the returns from reinvested coupon receipts increase (decrease). The fact that price risk and reinvestment risk move in opposite directions and are subject to the same influences offers a way to manage interest rate risk. The choice of the appropriate coupon and maturity bond to hold over a given investment holding period is often accommodated by means of duration.

To illustrate the general idea consider the case where the yield curve is flat but immediately after the bond is purchased the yield changes to some new value and stays there. In this case, the investor is faced with both reinvestment and price risk, and after m periods, the liability may not be covered by the accrued value of the coupons and the residual price of the bond. To focus on this problem, assume the firm has a future obligation of $\$F$ to be paid out in m periods. The current value of the liability is V , where

$$V_0(y) = F/(1+y)^m$$

To meet this obligation the firm reserves V_0 dollars, and uses this cash to purchase a coupon bond with maturity n, say. The value of this bond is

$$V_1(y) = \sum_{i=1}^{n} \frac{CF_i}{(1+y)^i}$$

Here, CF_i is the cash flow of the bond in period i, and, since the yield curve is flat, the yield y in both equations are the same. Moreover, by construction, the number of bonds purchased is such that $V_0(y) = V_1(y)$.

Now, assume a small shift in the yield curve occurs from y to $y+\Delta y$. The value of the liability changes from $V_0(y)$ to $V_0(y+\Delta y)$, while the value of the bond changes from $V_1(y)$ to $V_1(y+\Delta y)$. From the quadratic bond pricing approximating equation we have:

$$\begin{aligned} V_0(y+\Delta y) &\approx V_0(y) - D_m^0 V_0(y)\Delta y + \frac{1}{2}C^0 V_0(y)(\Delta y)^2 \\ V_1(y+\Delta y) &\approx V_1(y) - D_m^1 V_1(y)\Delta y + \frac{1}{2}C^1 V_1(y)(\Delta y)^2 \end{aligned}$$

For the liability to be immunized by the bond, their market values after the interest rate change should remain equal. Since $V_0(y) = V_1(y)$ this implies:

$$-D_m^0 V_0(y)\Delta y + \frac{1}{2}C^0 V_0(y)\Delta y)^2 = -D_m^1 V_1(y)\Delta y + \frac{1}{2}C^1 V_1(y)\Delta y)^2$$

For small interest rate changes the convexity adjustments are insignificant and the above equation reduces to

$$D_m^0 = D_m^1$$

That is the modified duration of the assets should be chosen to equal the modified duration of the liability.

In summary, we have shown that if the yield curve is flat, with a small parallel shift occurring after purchase of the bond, then, for the market value of the coupon bond to equal the market value of the future obligation, the duration of the bond selected should equal the holding period. Since, in our example the duration of the liability is m, the target bond portfolio should also have a duration of m.

The above result is really only approximate since the convexity adjustments were ignored. If the change in yields are large, this approximation may not hold very well. If a tighter immunization is required then the convexities should be matched as well. That is $C^0 = C^1$. For our particular problem, using the convexity equation, the convexity of the liability is given by

$$C^0(y) = \frac{m(m+1)F}{(1+y)^{2+m}V_0(y)}$$

Substituting for $V_0(y)$ and simplifying leads to

$$C^0(y) = \frac{m(m+1)}{(1+y)^2}$$

A bond having a modified duration of m, and satisfying the above equation will more precisely match the liability than an alternative bond that merely matches the duration.

Example

Consider a five-year bond that pays 12% annual coupons and yields 12%. Assume the investment horizon is four years. Since the modeified duration is four years, the bond is locally immunized. If rates fall immediately to 11% and stayed there over the four-year period, the drop from coupon reinvestment returns would be offset exactly by increases in the price of the bond.

Example

A \$1480 liability is due in 10 periods (5 years). The yield curve is currently flat at 4% per period (8% per year). With semiannual compounding the present value of this liability is $V_0(y) = 1480/(1.04)^{10} = \$1000$.

To immunize this liability an investor is considering purchasing the appropriate number of units of a bond that pays \$69.75 every period, has a face value of \$1,000, and matures in 14 periods. The current price of the bond is \$1,314.25, and its duration is 10 periods. Suppose \$1,000 (or 0.7609 units) of the bond were purchased. If the yield curve stays flat at 8%, and if all coupons are reinvested, then the value of this portfolio after 5 years would equal the liability of \$1480.

If, on the other hand, after buying the bond, the yield to maturity changed from 8 to 10 percent, and stayed there, then all bond prices would drop. However, the higher returns on coupon income would partially offset this price drop. The total accumulated value from holding one bond for 5 years (10 periods) is given by

$$\sum_{i=1}^{10} CF(1.05)^i + \sum_{i=1}^{4} \frac{CF}{(1.05)^i} + \frac{1000}{(1.05)^4} = \$1947.34$$

where $CF = \$69.75$ is the coupon payout in each period. Hence, the total value of 0.7609 units of the bond is $(0.7609)(1947.34) = \$1481.7$, which is sufficient to meet the \$1480.0 liability.

Figure 9.5 shows the accrued value of the portfolio for different changes in the yield curve.

Notice that regardless of the shift in the yield curve, price risk is offset by reinvestment risk and the accrued value from the portfolio always exceeds the liability.

Table 9.2 shows information on two alternative bonds, bonds, A and C.

Fig. 9.5 Accrued Value of the Portfolio for Different Parallel Shocks

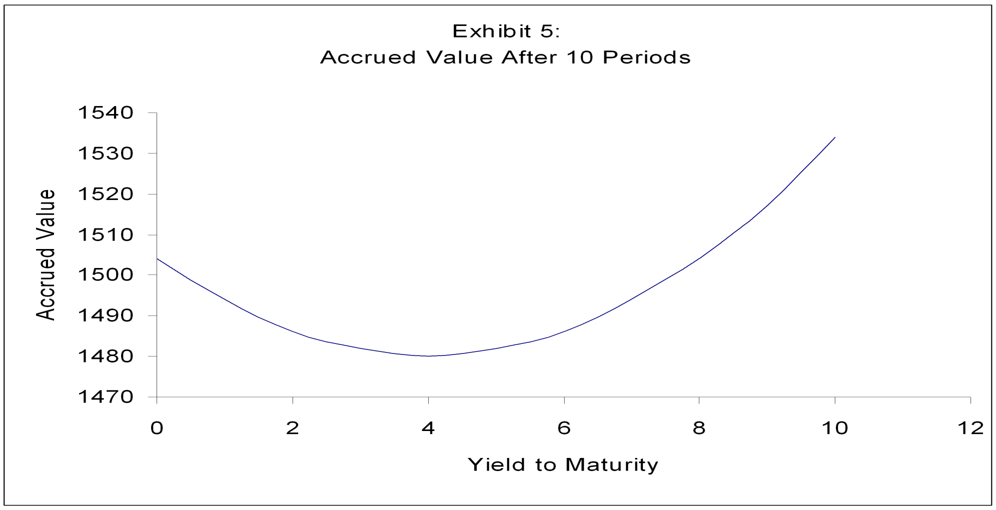


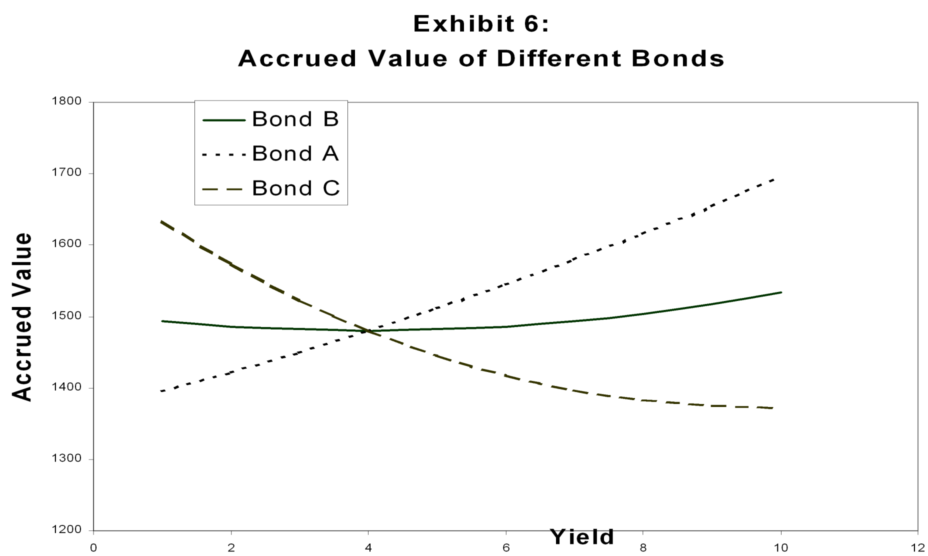
Table 9.2 Information on Two Bonds

	Bond A	Bond B	Bond C
Bond Price	1219	1314	1407
Amount of Bond	0.8203	0.7609	0.7103
Semiannual Yield	4%	4%	4%
Maturity (Periods)	10	14	20
Coupon	67	69.75	70
Face Value	1000	1000	1000
Duration	7.85	10	12.71

The previous analysis is repeated for these bonds and illustrated in Figure 9.6.

Notice that the total accumulated value after 5 years may not be sufficient to meet the liability. For example, by holding bond C, the investor is speculating that interest rates will decrease and that the loss of reinvestment income, obtained from lower yields will be more than offset by the accompanying price increase. Similarly, if Bond A (low duration) is held, the investor is speculating that interest rates will increase, and that the drop in price will be compensated by higher reinvestment income.

Fig. 9.6 Accrued Value



A portfolio of Bonds A and C could be constructed to have a duration equal to the duration of the liability (10 periods) and an initial value equal to the liability ($1000). Specifically, consider purchasing 0.4574 units of Bond A and 0.3143 units of Bond C. The value of this portfolio, V_p is

$$V_p = 0.4574(1218.99) + 0.3143(1407.71) = \$1000.$$

Moreover, the duration of the portfolio is easily computed. Specifically, the fraction of wealth,α_a, in Bond A is given by

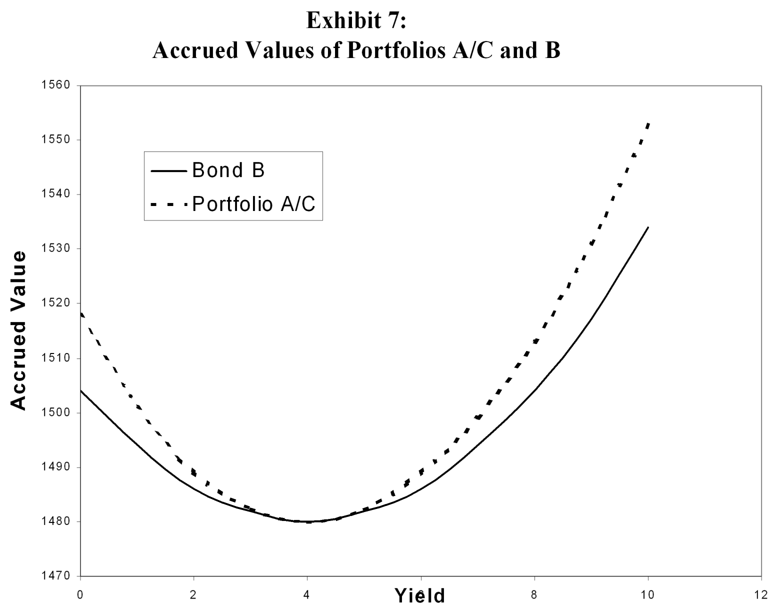
$$\alpha_a = \frac{0.4574 \times 1218.99}{1000} = 0.557.$$

Hence, the duration of the portfolio is D_p where

$$D_p = \alpha_a D_a + (1 - \alpha_a) D_c = 0.557 \times 7.85 + 0.442 \times 12.71 = 10.0\text{periods}$$

Figure **??** compares the accrued value of this portfolio to the earlier bond, Bond B. The above analysis reveals that many portfolios of bonds could be constructed to have the same duration as the targeted liability, and hence the portfolio that immunizes the liability by duration matching is not unique. In the above example the portfolio of bonds appears to be superior to Bond B because the portfolio's accrued terminal value is larger, for all yield shifts.

Fig. 9.7 Comparison of Accrued Values of Portfolio vs Bond B



Rather than match the convexity of the liability we might conclude that among all duration matched bond portfolios, the best portfolio is the one with the highest convexity. The appropriate portfolio of bonds that achieves maximum convexity can be obtained by solving the following linear programming problem.

$$\begin{array}{ll} Maximize & \sum_{i=1}^{K} \alpha_i C_i \\ \text{subject to} & \sum_{i=1}^{K} \alpha_i D_i = m \\ & \sum_{i=1}^{K} \alpha_i = 1 \end{array}$$

where α_i is the proportion of wealth allocated to bond i, and K represents the number of candidate bonds. A duration matched bond portfolio having maximum convexity has the property that the selected bonds usually have extreme durations and/or maturities. Such a portfolio is called a *barbell* portfolio since the pattern of cash flows reach their maximums at extreme time points.

Example

The yield curve is flat at 3%. A firm has a simple stream of liabilities that are shown in Figure 8.

Fig. 9.8 Stream of Liabilities

Exhibit 8

Liability Input Data

	Liability A	Liability B	Liability C
Periods to Maturity:	2	10	20
Interest Rate/Period:	3	0	3
Future Value:	1000	1000	1000
Number Held:	1	1	1

Liability Output Data

	Liability A	Liability B	Liability C	Portfolio
Present Value:	1000.00	744.09	1000.00	2744.09
Duration:	1.97	10.00	15.32	9.01
Convexity:	5.55	103.69	275.10	130.39

The firm wants to hedge these liabilities by purchasing an appropriate portfolio of bonds shown in Figure 9.9. The firm chooses a portfolio that matches the duration of the liability, but has a higher convexity. The specific portfolio, is outlined in Figure 9.10.

Figure 9.11 shows the accrued value of the asset and liability portfolios after 9 periods, corresponding to their durations. The values are computed under various assumptions on the size of a parallel shock that hits the yield curve at date 0.

Fig. 9.9

Exhibit 9

Bond Input Data

	Bond A	Bond B	Bond C
Periods to Maturity:	2	30	60
Coupon Rate/Period:	0	0	0
Face Value:	1000	1000	1000
number:	1	1	1

Bond Output Data

	Bond A	Bond B	Bond C	Portfolio
Present Value:	942.60	411.99	169.73	1524.32
Duration:	2.00	30.00	60.00	16.03
Convexity:	5.66	876.61	3449.96	624.57

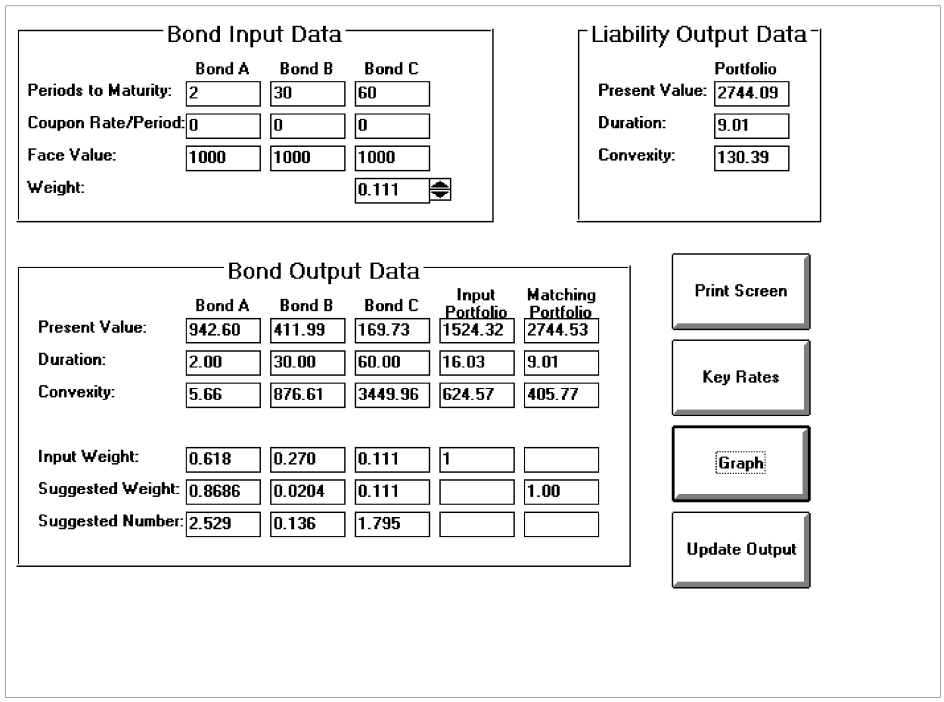
Notice that the value of the asset (liability) portfolio is at their lowest level if no shocks occur. If interest rates decrease, then while reinvested coupons earn a lower rate, this is more than offset by the increase in the sales price of the asset price of the bonds at date 9. Notice too, that because the asset portfolio is more convex, the value of the assets exceed the value of the liabilities.

Figure 9.12 shows the accrued value of the assets and liabilities after 5 periods. In this case, since the duration exceeds the holding period, there is a significant price risk. If interest rates increase in this period, the higher reinvestment rates on coupons will not offset the lower bond prices. While the asset portfolio dominates the liability portfolio, the exhibit clearly shows that price risk is not balanced against reinvestment risk over this period.

Figure 9.13 shows the accrued value of the two portfolios over 15 periods. Since duration is less than the holding period, we now see that both portfolios are suceptible to reinvetment risk. That is, if interest rates go down, the portfolios both depreciate. Again, due to convexity, the asset position appears to dominate, regardless of the parallel shock.

Fig. 9.10

Exhibit 10



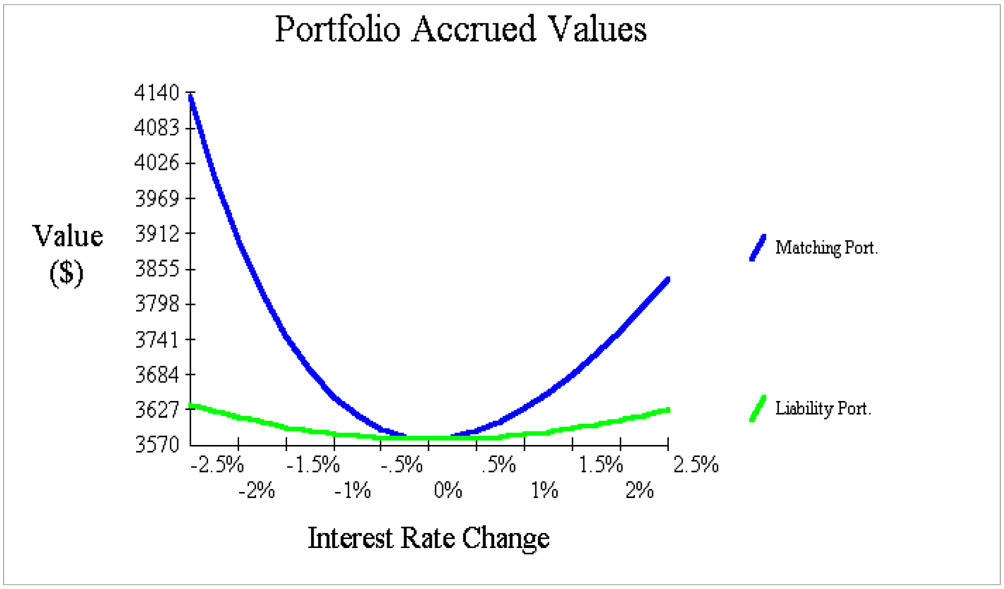
9.10 IMMUNIZATION AND TWIST RISK

The above linear programming problem produces a portfolio of bonds that is most convex, and hence most likely to produce surplus cash flows at the date of the liability. However, something should concern you. From the above analysis it looks like we can purchase high convex bonds, and sell low convex bonds with the same duration in such a way that our initial net investment is zero, but the value in the future will be guaranteed to be nonnegative in all states and positive in most. That is, there is a delightful riskless arbitrage strategy here. Where is the problem?

Recall that we assumed that the yield curve was flat and that the shock to the curve was a parallel shock. If this actually occurred, then indeed, there would be riskless arbitrage strategies. However, the yield curve may not

Fig. 9.11

Exhibit 11



undergo a parallel shift. One might surmize that the above portfolio strategy of buying high and selling low convexity bonds is very susceptible to shocks in the yield curve that are not parallel. This suspicion turns out to be correct.

The risk neglected by assuming parallel yield curve shifts is referred to as twist risk. We now consider a measure of twist risk. Exhibit 14 below shows the cash flows of two portfolios of bonds having the same duration as the target. Portfolio A is a barbell portfolio while portfolio B is closer to a bullet portfolio. As before the liability date is m.

While portfolio A has greater convexity than portfolio B, it is riskier. To see this, consider what happens if interest rates change in an arbitrary nonparallel way. Suppose short rates decline and long rates increase. The accrued values of A and B would be lower than the target value because of lower reinvestment rates and lower bond prices. However, the loss in A would be greater than B. First, lower reinvestment rates are experienced for longer periods with A, and second, the price risk at m is greater for the long bond in A than that in B.

Fig. 9.12

Exhibit 12

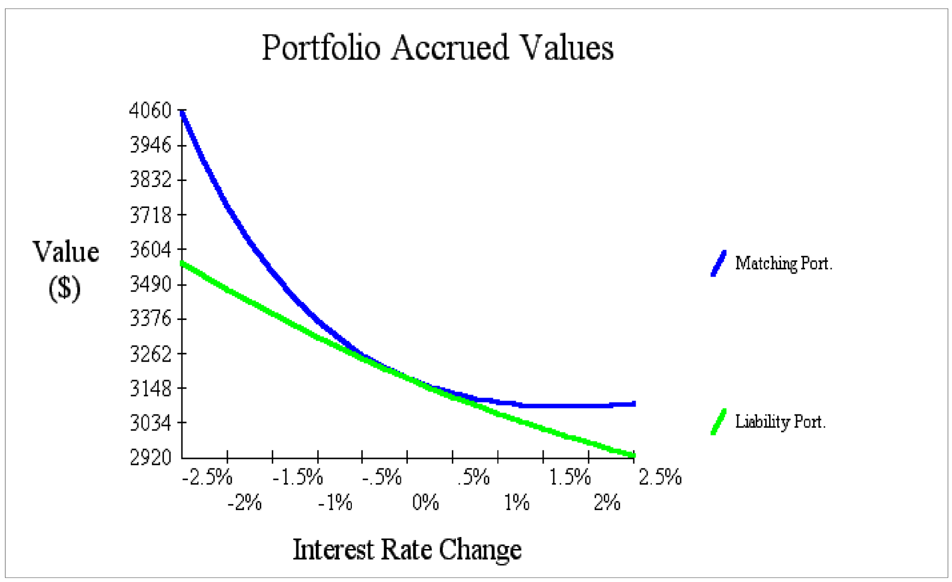
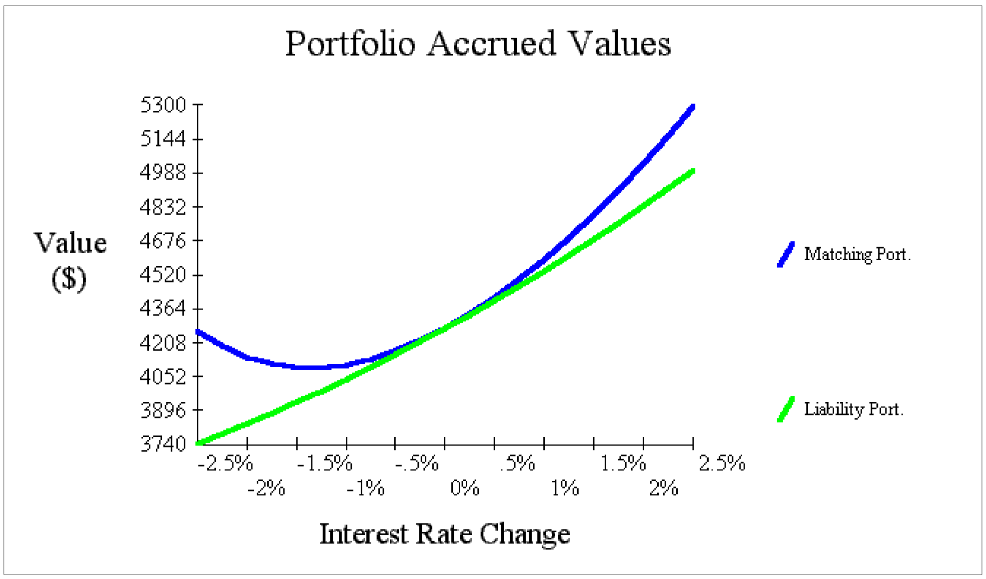


Fig. 9.13

Exhibit 13

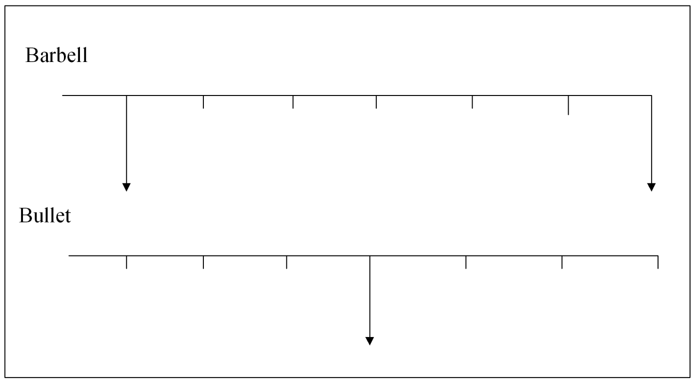


The bullet portfolio has much less exposure to a change in shape of the term structure. Indeed, if the bond portfolio immunizing the liability has no reinvestment risk, then the liability is completely immunized regardless of the shift in interest rate structure. When there are high dispersion of cash flows around the horizon date, as in the barbell portfolio, the portfolio is exposed to higher reinvestment risk, and hence greater immunization risk. Near bullet portfolios are therefore subject to less twist risk than barbell portfolios. A measure of twist or immunization risk is given by M^2 where

$$M^2 = \sum_{i=1}^{n} w_i(t-m)^2$$

where w_i is the present value of the i^{th} cash flow relative to the bond price, and m is the duration of the bond. Clearly, $M^2 = 0$ only if the weights are all zero except at date m where $W_1 = 1$. In this case the portfolio consists of a discount bond that perfectly immunizes the liability regardless of the types of shocks in the interest rate term structure. In general, the larger M^2 , the greater the variability of cash flows around the "target" date m. Fong and

Exhibit 14
Comparison of Barbell and Bullet Portfolios, with the same Duration.



Vasicek have shown that the lower the M^2 risk measure, the lower the risk is against nonparallel shifts in the yield curve. To minimize this twist risk an immunizing portfolio can be constructed by solving the following linear program:

$$\begin{array}{rl} Minimize & \sum_{i=1}^{K} \alpha_i M_i^2 \\ \text{subject to} & \sum_{i=1}^{K} \alpha_i D_i = m \\ & \sum_{i=1}^{K} \alpha_i = 1 \end{array}$$

The first constraint states that the duration must be m. The second constraint forces the sum of the fractions of wealth invested in the K bonds to add up to 1. Additional constraints, such as convexity and nonnegativity constraints, could be included.

Example

In our previous problem, if the shocks are not parallel shocks, then the value of the liabilities after 9 periods could exceed the value of the assets. A better hedge against interest rate shocks would be not only to duration match but also to convexity match.

9.11 DURATION AND CONVEXITY USING CONTINUOUSLY COMPOUNDED YIELDS TO MATURITY

In our analysis so far, we assumed the compounding interval was the same as the time between cash flows. Let y^* represent the annualized yield to maturity, and assume that compounding takes place k times a year. Then

$$\begin{aligned} B_0 &= \sum_{i=1}^{m} \frac{CF_i}{(1+y^*/k)^i} \\ D &= \sum_{i=1}^{m} (i/k) w_i \\ D_m &= \frac{D}{(1+y*/k)} \end{aligned}$$

where w_i is the contribution of the present value of the i^{th} cash flow to the bond price. Here B_0 is the bond price and D is the duration in years.

Notice that as $k \to \infty$, $y*/k \to 0$ and $D_m = D$. In this case the modification of duration to measure price sensitivity becomes unnecessary. Indeed, for continuously compounded yields, duration and modified duration are identical.

To see this recall

$$B = \sum_{i=1}^{m} CF_i \times e^{-t_i y}$$

Hence

$$\frac{dB}{dy} = -\sum_{i=1}^{m} t_i CF_i \times e^{-t_i y}$$

and

$$\frac{dB}{B} = (\sum_{i=1}^{m} i w_i) dy$$

where

$$w_i = \frac{e^{-y t_i} CF_i}{B} = \frac{P(0,t_i) CF_i}{B}$$

9.12 DURATION WITH NONFLAT TERM STRUCTURES

So far we have assumed that the term structure of interest rates is flat at y. Specifically, $y(0,t) = y$. Then, when a shock ocurrs to the term structure, all yields move the same amount. Hence after the shock all yields to maturity are $y + \Delta y$. We have seen that if such shocks were the only shocks to occur, then there would be arbitrage opportunities. Hence limiting yield curves to be flat, and shocks to be parallel, is unreasonable. Actually, for the analysis in this chapter, we do not need yield curves to be flat. What we do require is the shock to be the same for all yields. In particular, using continuously compounded returns, we have.

$$B(0) = \sum_{i=1}^{m} CF_i \times e^{-y(0,t_i)t_i}$$

where $y(0, t_i)$ is the actual continuously compounded yield to maturity for the time period $[0, t_i]$ where t_i is the time in years to the *ith* cash flow. Then we could assume all yields increase by Δy. Then, the new bond price would be

$$\sum_{i=1}^{m} CF_i \times e^{-(y(0,t_i)+\Delta y)t_i}$$

The duration measure can then be computed by differentiating this equation with respect to Δy. This leads to

$$D_m = \sum_{i=1}^{m} t_i \Delta t w_i$$

where $w_i = \frac{CF_i P(0,t_i)}{B_0}$

Example

To Be Done

9.13 DURATION DRIFT AND DYNAMIC IMMUNIZATION STRATEGIES

As time advances, the duration of the portfolio changes, and the holding period diminishes. Unfortunately, these two values will not decline at the same rate. Duration, in fact, decreases at a slower rate, a process referred to as duration drift. Hence a portfolio of coupon bonds with duration matching the original liability period, m, will, after a certain amount of time, t say, have a duration exceeding the target value, $m - t$. This implies that to maintain an immunized position, the portfolio needs to be periodically readjusted such that its duration is reset to the time remaining to the liability payment.

The duration procedures described in this chapter have for the most part assumed yield curves to be flat and shocks to the yield curve have been restricted to parallel shifts. In practice, yield curves do not behave this way. Indeed it is unreasonable to assume yields-to-maturity on different assets will change by the same amount. First, yields to maturity are complex averages of the underlying spot rates. A given shift in the spot rate curve will result in the yields to maturity on different assets changing by differing amounts. Second, yields for different maturities are imperfectly correlated. If short term rates increase by 1%, long term rates typically move by less than 1%. Indeed the short rate and long rate could move in different directions, causing the yield curve to increase (decrease) in steepness, for example. This twisting shape in the yield curve is not explicitly considered in the previous models. It suggests that the yield curve responds to more than one factor. With more than one factor causing shocks to the yield curve, more complex duration models need to be established. Ideally, these models should be based on more realistic models of interest rate behavior. A significant body of research has been devoted to this problem, and many alternative duration measures have been constructed.[3]

9.14 DURATION OF FLOATING RATE NOTES.

Consider a floating rate note that pays according to 6 month LIBOR every 6 months for n years. LIBOR is determined at the beginning of each period and paid at the end of the period. Let $\ell[t_i, t_{i+1}]$ represent the Libor rate at date t_i for the time period $[t_i, t_{i+1}]$. The cash flow of $N\ell[t_i, t_{i+1}]\Delta t_i$ occurs at date t_{i+1}. We shall assume the face value N is \$100. To convert 6 month LIBOR into a semiannually compounded rate, we have

$$\frac{y}{2} = \ell[t_i, t_{i+1}] \times \frac{\text{Days in period}}{360}$$

We have seen that the price of a floating rate note at date 0, is given by:

$$V_{FLOAT} = 100P(0, t_0) = \frac{100}{(1 + y/2)^p}$$

where p is the fraction of a six month period remaining to the next reset date.

[3]We shall consider some extensions to the theory in the next chapter.

Alternative Development of the FRN Pricing Equation

Assume the maturity of the FRN was 6 months. If the bond equivalent yield is y, we have:

$$V_{FLOAT} = \frac{100(1 + y/2)}{1 + y/2} = 100$$

Now assume the bond has 12 months to go. We know that after 6 months the bond will trade at par. Hence the value of the bond is the present value of $100 plus the present value of the interest over a reset period. That is:

$$V_{FLOAT} = \frac{100(1 + y/2)}{1 + y/2} = 100$$

Repeating this arguement recursively, reveals that at a reset date the FRN is always set at its face value.

Now consider what happens between reset dates. The final cash flow is given by $100(1 + y^*/2)$ where y^* was determined at the previous rest point. The price is therefore given by

$$V_{FLOAT} = \frac{100(1 + y^*/2)}{(1 + y/2)^p}$$

where p is the fraction of a six month period remaining to the cash flow, and y is the bond equivalent yield associated with the current LIBOR rate over the time to the next reset date.

The Duration of the FRN is therefore given by differentiating the above equation with respect to y. The result is:

$$D_{FLOAT} = \frac{p/2}{1 + y/2}$$

The modified duration of FRNs therefore range from 0 to 0.5 years. FRNs hence behave like fixed rate notes with maturities with less than six months remaining to maturity.

9.15 DURATION FOR INTEREST RATE SWAPS

An interest rate swap can be viewed as a long position in a fixed rate note and a short position in a floating rate note. We can compute the duration of each of the legs separately, but we cannot really compute the duration of the swap, since at the initiation date the value of the swap is zero. In a portfolio context, however, the swap can be handled quite effectively.

In practice, it may make more sense to investigate the absolute change in value of the swap to a yield curve shock. For example, the PVBP of an interest rate swap can be easily obtained.

Example

To be done

9.16 INVERSE FLOATERS

An inverse floating rate note is a FRN, where the rate varies inversely with the index rate, such as 6 month Libor. In particular,

$$Rate_{t_i} = k - \ell[t_i, t_{i+1}]$$

A floor of zero is placed on this rate to ensure that the number cannot go negative.

Example

A dealer purchases 100m dollars of a fixed rate bond, that pays 10% coupons semiannually, and place it in trust. The trust then issues a 50m dollar floater and a 50m dollar inverse floater with the floater linked to six month Libor. The rate on the inverse floater is:

$$10\% - \text{Libor}.$$

At each reset/coupon date the trust receives $100 \times 0.10/2 = 0.5$ million dollars. In addition the trust is responsible for paying $50y^*/2$ million dollars on the floaters and $50(0.10 - y^*)/2$ million dollars on the inverse floaters. Here y^* represents the 6 month Libor rate. The net sum the trust is responsible for is $50 \times 0.05 = 0.25$ million dollars.

The buyer of an inverse floater can be viewed as long the fixed bond and short the floater. Ignoring the floor, we therefore have:

$$V_{INVERSEFLOAT} = V_{FIXED} - V_{FLOAT}$$

The duration of an inverse floater can now easily be determined. Let

$$\begin{aligned} w_{FIXED} &= \frac{V_{FIXED}}{V_{INVERSE\ FLOAT}} \\ w_{FLOAT} &= \frac{V_{FLOAT}}{V_{INVERSE\ FLOAT}}. \end{aligned}$$

Then, the duration of the inverse floater is:

$$D_{INVERSEFLOATER} = w_{FLOAT} D_{FLOAT} + w_{FIXED} D_{FIXED}$$

9.17 CONCLUSION

This chapter has reviewed the basic concepts of risk management measures in the bond market. The most common measures of the sensitivity of default free bonds to interest rate changes is captured by modified duration and convexity. The assumption here is that when shocks occur in the yield curve they are parallel shocks. Applications of these measures were provided. In particular we investigated how portfolio managers can use duration and convexity to immunize debt obligations.

9.18 REFERENCES

This chapter draws very heavily from the presentation of duration in the textbook Numerical Methods in Finance, by Simon Benninga. This book and accompanying software are highly recommended. Another excellent discussion of duration is given by Dunetz and Mahoney. For a comprehensive treatment of duration, the text by Bierwag is recommended.

Benninga, S. "Numerical Methods in Finance", MIT Press, 1989.

Bierwag, G.O., 1977, "Immunization, Duration, and the Term Structure of Interest Rates", Journal of Financial and Quantitative Analysis, Vol. 12, 725-741.

Bierwag, G.O., 1987, Duration Analysis: Managing Interest Rate Risk, Cambridge, MA: Ballinger.

Bierwag, G.O., I. Fooladi and G.S. Roberts, 1993, "Designing an Immunized Portfolio: Is M-squared the Key?", Journal of Banking and Finance, 17, 1147-1170.

Bierwag, G.O., G.G. Kaufman, and C. Khang, 1978, "Duration and Bond Portfolio Analysis: An Overview", Journal of Financial and Quantitative Analysis, 13, 671-685.

Bierwag, G.O., G.G. Kaufman, C. Latta and G.S. Roberts, 1987, "Usefulness of Duration in Bond Portfolio Management: Response to Critics", Journal of Portfolio Management.

Bierwag, G.O, G.C. Kaufman, R. Schweitzer and A. Toevs, 1981, "The Art and Risk Management in Bond Portfolios", Journal of Portfolio Management, Spring, 27-36.

Bierwag, G.O., G.C. Kaufman and A. Toevs, 1982, "Single-Factor Duration Models in a General Equilibrium Framework", Journal of Finance, 37, 325-338.

Chambers, D.R., 1984, "An Immunization Strategy for Future Contracts on Government Securities", The Journal of Future Markets, 4, 173-188.

Chambers, D.R. and W. Carleton, "A Generalized Approach to Duration", Research in Finance, Vol. 7, 1988, 163-181, JAI Press Inc.

Chua, J., 1984, "A Closed-Form Formula for Calculating Bond Duration", Financial Analysts Journal, May/June, 76-78, and G.C. Babcock, "Druation as a Weighted Average of Two Factors", Financial Analysts Journal, March/April 1985, p.75.

Dunetz, M. and J. Mahoney, "Using Duration and Convexity in the Analysis of Callable Bonds", Financial Analysts Journal, May 1988, 53-72.

Fong, H.G. and F.J. Fabozzi, 1985, Fixed Income Portfolio Management, Homewood, IL: Dow-Jones Irwin.

Fong, H.G. and O. Vasicek, "The Tradeoff Between Return and Risk in Immunized Portfolios", Financial Analysts Journal, September-October 1983, 73-78.

Fong, W.J. and G.A. Vasicek, 1983, Return Maximization for Immunized Portfolios", in G.O. Bierwag, G. Kaufman and A. Toevs, eds., Innovations in Bond Portfolio Management, Greenwich, CT: JAI Press, 227-238.

Ingersoll, J.E., J. Skelton and R.L. Weil, 1978, "Duration Forty Years Later", Journal of Financial and Quantitative Analysis, 13, 627-652.

Livingston, M., 1979, "Measuring bond Price Volatility", Journal of Financial and Quantitative Analysis, 14, 343-349.

Macaulay, F.R., 1938, Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the U.S. Since 1856, New York: National Bureau of Economic Research.

Nawalkha, S.K. and N.J. Lacey, 1988, "Closed-Form Solutions of Higher-Order Duration Measures", Financial Analysts Journal, November-December, 82-84.

Prisman, E.Z. and M.R. Shores, 1988, "Duration Measures for Specific Term Structure Estimates and Applications to Bond Portfolio Immunization", Journal of Banking and Finance, 12, 493-504.

9.19 EXERCISES

1. The term structure is flat at 8%. Consider an 8% coupon bond with semiannual payouts that matures in 10 years. If yields increased by 1 basis point ($y = 8.01\%$) what would be the effect on price? If the yield curve was flat at 9% and increased by 1 basis point, would the price effect be bigger or smaller. Explain.

2. A bond has 2-years to maturity, pays semiannual coupons, and a face value of $1,000. The coupon is 7%, and the yield-to-maturity is 8%.

   (a) Compute the price of the bond.

   (b) Compute the duration, and modified duration of the bond.

   (c) Compute the convexity of the bond.

3. Assume the yield on the bond changes from 8% to 8.05%.

   (a) Using the bond pricing equation, compute the new price of the bond, and then establish the change in the bond price.

   (b) Using the linear approximation, compute the change in the bond price.

   (c) Using the quadratic approximation, compute the change in the bond price.

   (d) Compare the answers in (b) and (c) to that in (a) and draw conclusions.

4. Repeat problem (2), but this time assume the yield changes from 8 to 10%.

5. A discount bond is available with a maturity of 6 years. The annualized yield-to-maturity of this bond is 7% (computed on a semi annualized basis) and the face value is $1000.

   (a) Compute the price of this bond.

   (b) Compute the duration, modified duration and convexity of this bond.

6. A trader has an obligation of \$10m due in 4 years. The trader decides to purchase units of the bond in question (1) and the bond in question (3), and wants to choose the portfolio that has a duration equal to that of the liability. Establish the required portfolio.

7. A 8 year bond has annual coupons. The coupon is 10%, the face value is \$1000, and the current yield curve is flat at 10%.

   (a) Compute the duration of the bond.

   (b) Assume the yield curve increased from 10% to 10.5%, and remained unchanged for two years, at which time the bond was sold. Assume all coupons were reinvested. What would be the total accrued value of the account.

   (c) Another coupon bond with a duration of two years was available. Assume the number of units purchased of this bond were chosen so that the initial investment was equal to the inital value of the above bond. After two years, would the accrued value of the account be more insulated from an immediate parallel shift in the yield curve. Explain.

8. Assume the yield curve is flat at 10%. (y = 10%) Consider the following 3 bonds:

Bond	A	B	C
Maturity (Years)	2	5	8
Coupon	8%	10%	2%
Face Value	1000	1000	1000

   (a) Compute the prices of the 3 bonds.

   (b) Compute the duration, modified duration and convexity of the 3 bonds.

   (c) Use bonds A and B to construct a portfolio that has a duration of 3 years.

   (d) Use bonds A and C to construct a portfolio that has a duration of 3 years.

   (e) Of the two portfolios constructed in (c) and (d), which one is more convex. If the trader wanted a duration of 3 years to immunize an obligation due in 3 years, which of the above two portfolios would you recommend? Explain.

(f) Of all the portfolios of bonds A, B and C, find the portfolio that maximizes convexity. Assume the trader is not allowed to sell bonds short. That is, solve the linear programming problem for the maximum convexity problem but include in your constraints, the fact that all the weights of the portfolio must be nonnegative.

(g) Compute the twist risk for each bond, and then identify the minimum twist risk portfolio that has a duration of 3 years. Again, in your linear programming formulation, assume the trader is only concerned with portfolio weights that are non-negative.

(h) Compute the portfolio that maximizes twist risk, subject to the duration and nonnegativity constraints. Compare this portfolio to that obtained in (f).

Appendix
Derivation of Modified Duration and Convexity

Differentiating the bond pricing equation leads to

$$dB/dy = -\sum_{t=1}^{m} \frac{tCF_t}{(1+y)^{t+1}}$$

Dividing both sides by the bond price yields

$$\frac{dB}{dy}\frac{1}{B} = \frac{-\sum_{t=1}^{m} \frac{tCF_t}{(1+y)^t}}{B(1+y)}$$

or

$$\frac{dB}{dy}\frac{1}{B} = -D\frac{1}{1+y}$$

which leads to the duration equation.

Differentiating the bond pricing equation two times, leads to:

$$\frac{d^2B}{dy^2} = \sum_{t=1}^{m} \frac{t(t+1)CF_t}{(1+y)^{t+2}}$$

from which the convexity equation follows.

Finally, use Taylor's expansion of B(y):

$$B(y+\Delta y) \approx B(y) + \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}(\Delta y)^2$$

Now substituting in the modified duration and convexity exprtessions leads to the quadratic approximation equation.