Option Models for Bonds and Interest Rate Claims

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Learning Objectives

- We want to be able to price any fixed income derivative product using a binomial lattice.
- When we use the lattice to price a straight bond, we would like the price to equal the observed price!
  - That is, prices of liquid claims (like coupon bonds) should automatically equal the observed prices
  - That way, options (for example) are set relative to observed prices!

Learning Objectives

- We also would want prices to be set such that there are no arbitrage opportunities.
- Examples of claims that we would price
  - Calls on Bonds
  - Calls on FRAs
  - Calls on ED Futures
  - Futures vs Forwards
  - Structured Products (callable, putable, callable-putable bonds)
  - Options on swaps (swaptions), caps, floors, etc

Learning Objectives

- There are several difficulties
  - (1) How do we generate prices that admit no arbitrage.
  - (2) How do we generate prices on a lattice that match todays discount function.
    - Obviously if we match the discount function, we also match the yield curve, the forward curve, the swap curve, the par curve!
- There are many ways in doing this.
  - That is there are many models available, most of which use lattice methods….

- So, today, we will learn:
  - How to set up a set of prices that are internally consistent. That is the set of prices are arbitrage free!
  - How to set up a set of prices that are externally consistent. That is the set of bond prices satisfy certain constraints.
    - » They match observables
    - » Volatilities have some desirable properties.
    - » Other properties (some derivative products match observables!)

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Modeling Uncertainty of the Term Structure

- How do you generate a set of internally consistent bond prices.
  - No arbitrage opportunities with the bonds.
  - For example, if we knew that a parallel shock to the yield curve would hit, either up or down, then we could make money without taking any risk!
  - Therefore parallel shocks are not consistent with no arbitrage
  - What dynamics are consistent with no arbitrage?

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The dynamics of the yield curve

Suppose the initial yield curve is flat at 10%. In the next period, the term structure is either flat at 12% or flat at 8%. Is such a representation of yield curve dynamics fesible?

Bond prices can easily be computed under the term structures at date 0 and at date 1. For example, at date 0, the price of a 1-year bond is $P(0,1) = e^{-0.10} = 0.9048374$.

In year 1, there are two possible bond prices. Let $P^{+}(1,1+t) = e^{-0.12\times t}$ be the price of a bond that has t years to maturity, and let $P^{-}(1,1+t) = e^{-0.08\times t}$ be the prices if rates decrease.

The Dynamics of the Yield Curve

- The yield curve is flat at 10%
- Next period it is flat at 12% or flat at 8%
- $P(0, t) = e^{-0.10t}$; $P(0,1) = e^{-0.10} = 0.9048374$,
- $P^{+}(1,1+t) = e^{-0.12t}$ or $P^{-}(1,1+t) = e^{-0.08t}$
- Is it possible to set up a zero cost portfolio at date 0 that has the property that it will make money in the up state and in the down state.

The Dynamics of the Yield Curve

•Under this structure an arbitrage profit can be generated.

Buy n_1 bonds that mature at date 1
Buy n_3 bonds that mature at date 3
Sell n_2 bonds that mature at date 2

Choose $n_1 = 0.452509559$
$n_2 = 1$
$n_3 = 0.552474949$

Initial Value	=	0
Value in upstate	=	0.0001809
Value in downstate	=	0.0001809

What does this example imply?

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Internally Consistent Prices

This example very clearly shows that if yield curves are known to be flat in period 1, at two values, then a flat yield curve at date 0 leads to a set of prices that are not internally consistent. The yield curve at date 0 in this example, cannot be flat, and discount bonds cannot be all priced to yield 10%. This example, raises the issue of how "sensible" prices of contingent claims can be established once future possible yield curves have been identified.

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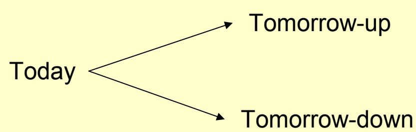
Types of Bonds

- Straight Bonds
- Callable Bonds
- Putable Bonds
- Floating Rate Notes
- Floating Rate Notes with Caps.
- Step-up callable notes
- Extendable Bonds
- Structured Products…………..

Interest Rate Models

- Description of how interest rates change over time.
- Single factor models
  - Focus on the behavior of the short rate, which is the single factor
  - Once the short rate dynamics are given, somehow we can construct the values of all other rates.
- Two factor Models
  - Focus on the behavior of a short rate and a long rate. Given these two values, we try and get all other yields.

•States are scenarios.



State Contingent Claims: Cash flow depends on state
Assets are collections of state contingent cash flows.

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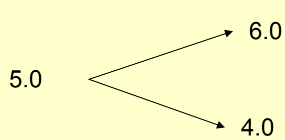
Interest rate scenarios

•States are scenarios.



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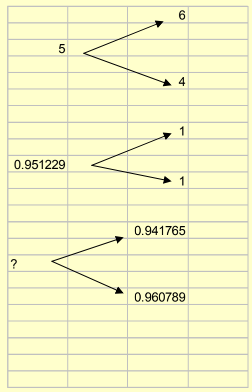
Pricing a Two period Bond?



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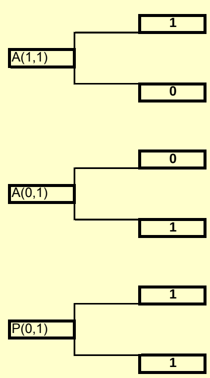
State Prices



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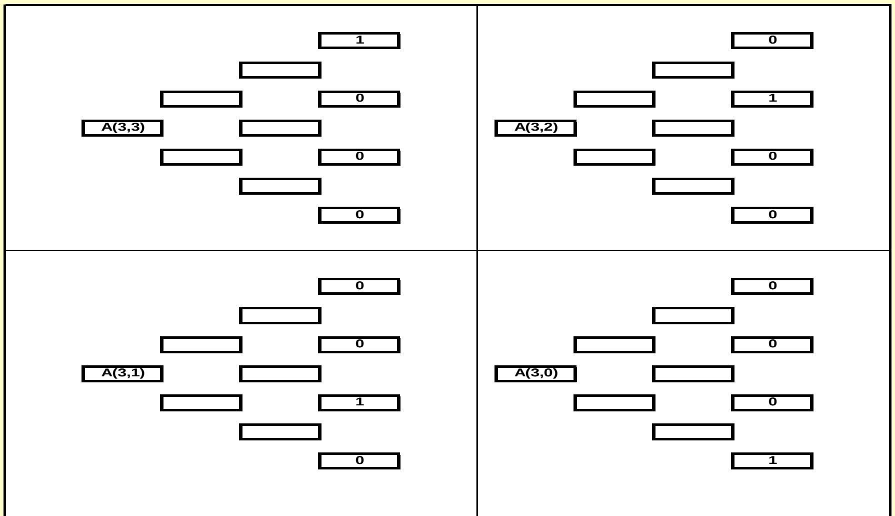

P(0,3)= A(3,0) + A(3,1) + A(3,2)+ A(3,3)

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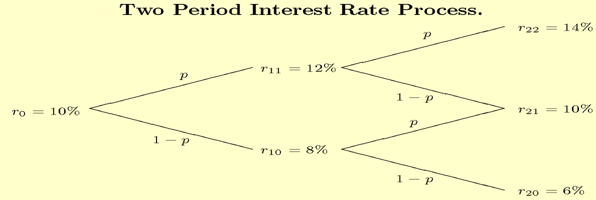
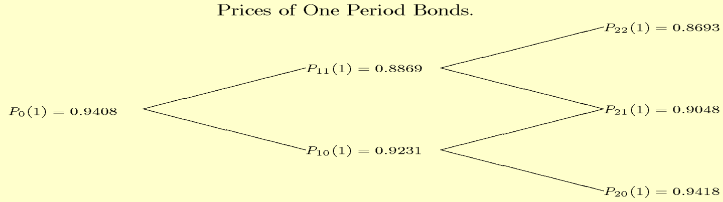
Two Period Interest Rate Process.



Prices of One Period Bonds.



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Local Expectations Hypothesis

Assume the true probability of an up jump is 0.5 and that the local expectation hypothesis holds.[1] Let $P_{ij}(m)$ represent the price of an m-period bond at node (i,j). Then,

$$
\begin{aligned}
P_{11}(2) &= [0.5P_{22}(1) + 0.5P_{21}(1)]P_{11}(1) = 0.7868 \\
P_{10}(2) &= [0.5P_{21}(1) + 0.5P_{20}(1)]P_{11}(1) = 0.8523 \\
P_0(2) &= [0.5P_{11}(1) + 0.5P_{10}(1)]P_0(1) = 0.8514 \\
P_0(3) &= [0.5P_{11}(2) + 0.5P_{10}(2)]P_0(1) = 0.7710
\end{aligned}
$$

The forward price $F_0[2,3] = P_0(3)/P_0(2) = 0.9056$. This implies that the forward rate $f_0[2,3] = 9.92\%$. Notice that this is not equal to the expected future spot rate which is 10%.[2]

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No arbitrage condition

1. *If a set of prices is free of arbitrage opportunities, then it has to be the case that there exists a set of risk neutral probabilities under which the local expectations hypothesis holds.*
2. *If no probability distribution can be constructed such that all market prices can be computed as their simple discounted expected value, then there are arbitrage free opportunities among the existing prices.*
3. *If there is a unique risk neutral probability distribution, then the market is said to be complete.*

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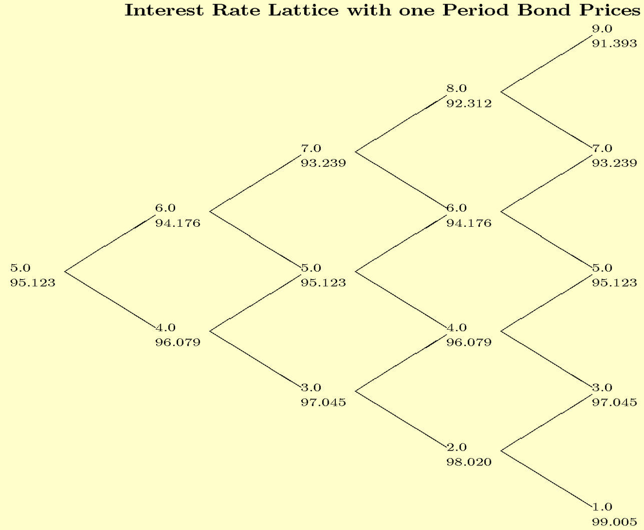
Internally Consistent Pricing

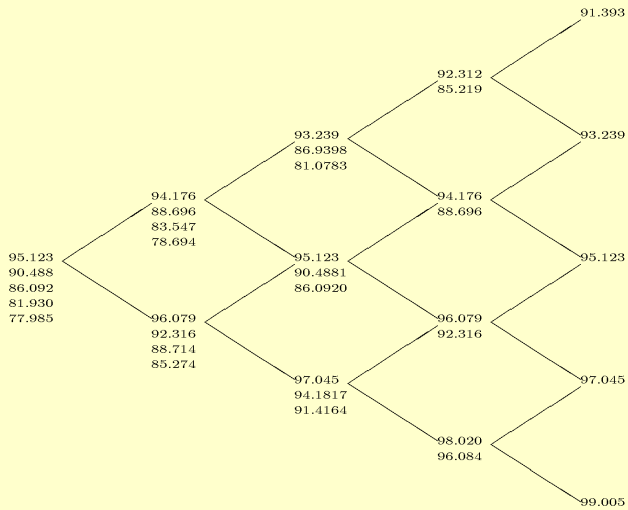
A set of internally consistent prices can be generated using the following three steps:

- Generate a "lattice" of potential interest rates.
- Choose "risk neutral" transition probabilities for interest rate moves on the lattice.
- Price claims on the lattice using the local expectations hypothesis.

Interest Rate Lattice with one Period Bond Prices



Bond Prices



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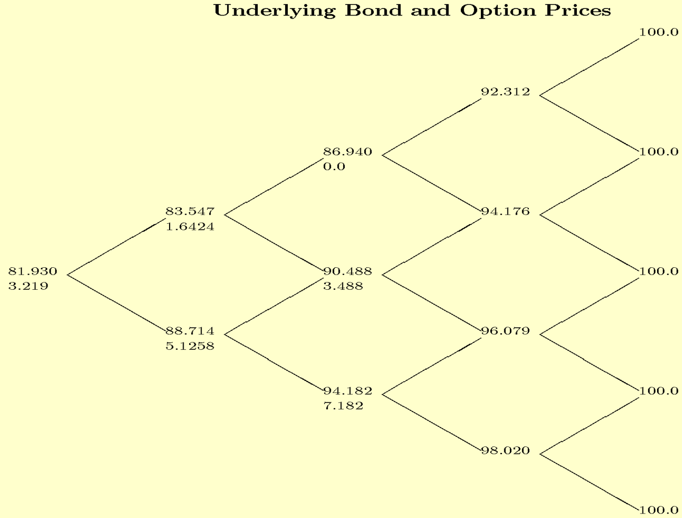
Underlying Bond and Option Prices



Two period call with strike 87

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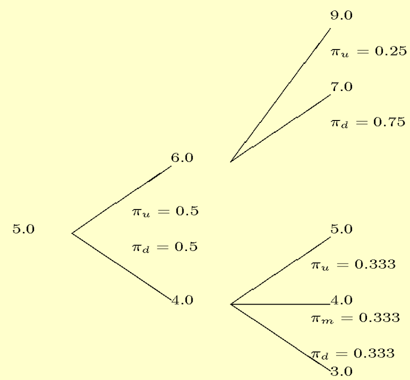
Interest Rate Lattice



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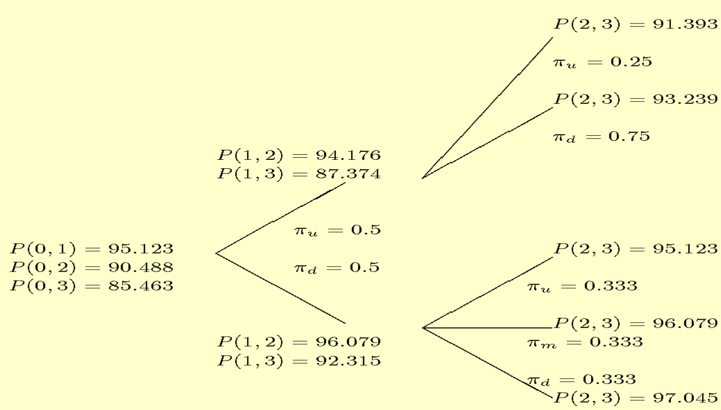
Selected Bond Prices[3]



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Externally Consistent Prices

- Notice that once we specify the lattice and probabilities, the discount function can be produced.
- In most applications we know the discount function.
- We want to build a lattice with the property that the resulting discount function matches observed strip prices.

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Externally Consistent Lattice Models

- As an example, assume the yield curve is flat at 10.536% (continuously compounded)
- Then, the discount function for five years is given by

P(1)=	0.9
P(2)=	0.81
P(3)=	0.729
P(4)=	0.6561
P(5)	0.59049

How can you build a lattice such that the bond prices match these observed prices.

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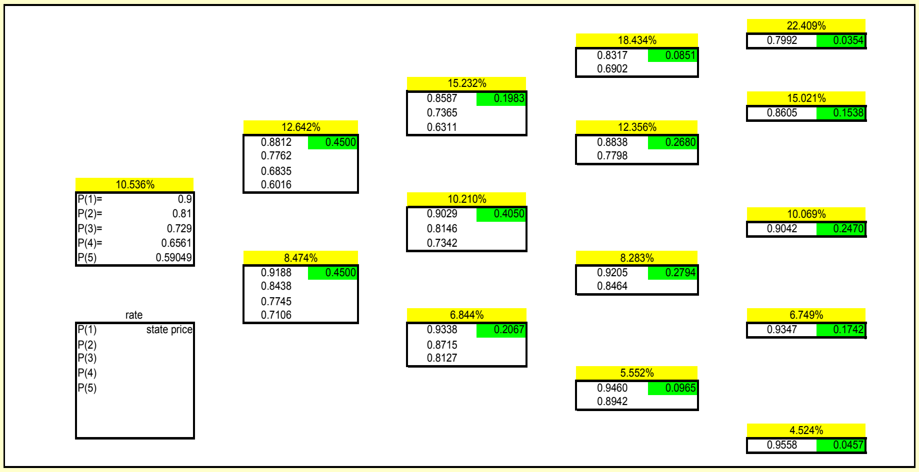
Here is a lattice that accomplishes this!



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Internally and Externally Consistent Pricing

- We can keep guessing a lattice and risk neutral probabilities, until our theoretical strip prices match actual strip prices
- This is not feasible.
- Lets be smart!
  - We will assume all risk neutral probabilities are 0.5, and use only binomial lattices.
  - We will slowly build a lattice that matches strip prices in each period.

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- We have the starting interest rate, r(0) = 0.09.
- Assume the low interest rate in the next period is r.
- The high interest rate in the next period is ru
- Here u = $e^{2\sigma}$ where the volatility σ=0.20. The bigger the volatility the greater the gap.
- Now, given the two interest rates, we can compute the two one period discount factors and the date 0 two period price. Rather than guess and check we solve:

$$P(0,2) = P(0,1)[0.5e^{-r} + 0.5e^{-2r\sigma}].$$

  for r
- The solution is r = 0.07236;
- Hence the upper interest rate is ru = 0.1079

We now can compute the two discount factors in period 1. They are 0.93019 for the low state and 0.8977 for the high state.

We now guess the lowest interest rate in period 2. Let r_2 represent this rate. The three possible interest rates are r_2, r_2u_2, and $r_2u_2^2$, where $u_2 = e^{2\sigma_2}$, and σ_2 is the volatility of the one period rate in the second period. In our example, the spot rate volatility stays constant at 20%, so $\sigma_2 = 0.20$. We want to choose r_2 such that the bond price, computed on the lattice, $P(0,3)$ equals its market value.

State Prices

To compute r_2 it is helpful to price each state security.
State price for High state in Period 1 is A(1,1) = ?
State Price for Low state in Period 1 is A(1,0) = ?

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State Prices

To compute r_2 it is helpful to price each state security.
State price for High state in Period 1 is A(1,1) = P(0,1) 0.5 = 0.4569
State Price for Low state in Period 1 is A(1,0) = P(0,1) 0.5 = 0.4569

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- A(1,0) = d(0,1)[0.5(0) + 0.5(1)] = d(0,1) 0.5
- A(1,1) = d(0,1) [0.5(1) + 0.5(0)] = d(0,1) 0.5

- We have the state prices for period 1.
- We have computed the interest rates at period 1
- We complete the stage by computing the next state prices. That is compute A(2,0), A(2,1) and A(2,2).

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- A(2,0) = ?
- Standing at node (1,0) what is the price of a claim that pays $1 in node (2,0) and zero elsewhere?
- The answer is 0.5 1 d(1,0) dollars.
- So viewed at date 0, the value is 0.5 1d(1,0) A(1,0)
- Hence A(2,0) = 0.5d(1,0)A(1,0)

- A(2,2)=?
- You get to node (2,2) through (1,1). So the answer is A(1,1) times something!
- Standing at (1,1) what is the value of 1 dollar in node (2,2)?
- The answer is 1 (0.5) d(1,1)
- So A(2,2) = A(1,1)d(1,1)0.5

- What about A(2,1)=?

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- There are two paths to node (2,1). One through (1,1) and one through node (1,0).
- Hence
- A(2,1) = A(1,1) times something +
  A(1,0) times something

A(2,1) = A(1,1)[0.5 d(1,1)] + A(1,0)[0.5d(1,0)]

So to get the date 0 state prices for period 2, all you need are the state prices for period 1 and the interest rates (discount factors) for period 1!

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- A(2,0) = A(1,0)[0.5d(1,0)] = 0.2125
- A(2,1) = A(1,1)[0.5d(1,1)) + A(1,0)(0.5d(1,0)]=0.4176
- A(2,2) = A(1,1)[0.5d(1,1)] = 0.2051

- The sum of these gives P(0,2)

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- Given the interest rates and discount factors in year 1 and the state prices for year two completes a stage of the process
- We now repeat these steps sequentially.

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- P(0,3) = A(2,0)e^{-r} +A(2,1)e^{-ru} +A(2,2)e^{-ruu}
- 0.7634 = 0.2125e^{-r} + 0.4176e^{-ru} +0.2051e^{-ruu}
- $u_2 = e^{-2\sigma_2} = 1.4918.$

Solving we obtain:
r = r(2) = 0.05839
This gives us the three one period bond prices. d(2,0),d(2,1) and d(2,2)

Then:
A(3,0) = 0.5A(2,0)d(2,0)
A(3,1) = 0.5A(2,0)d(2,0)+0.5A(2,1)d(2,1)
A(3,2) = 0.5A(2,1)d(2,1)+0.5A(2,2)d(2,2)
A(3,3) = 0.5A(2,2)d(2,2)

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Summary of the main idea

- Assume a value for the lowest rate in period 1, r
- Compute the value $re^{2\sigma}$
- Use these two rates to discount the cash flows of a two period discount bond.
- If the PV exceeds the observed market price, your guess was too low; increase r and try again
- If the PV is lower than the observed market price, your guess was too high; decrease r and try again
- Repeat until the two are equal.
- Go to period 2. Pick any value for the lowest state, r say. The interest rate levels are then $r, re^{2\sigma}, re^{4\sigma}$
- Repeat the iteration process until you find an r such that the three period bond price matches.
- Guess the lowest value for period 3, r say. The interest rate levels are then $r, re^{2\sigma}, re^{4\sigma}, re^{6\sigma}$
- Repeat the iteration process until you find an r such that the four period bond price matches

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- Using state prices makes the problem easier to solve. So after you have found the interest rates, for a time period slice, compute the next set of state prices.
- Use these state prices, together with the the next period $r, re^{2\sigma}$…..etc to get the bond equation that you have to solve for r.

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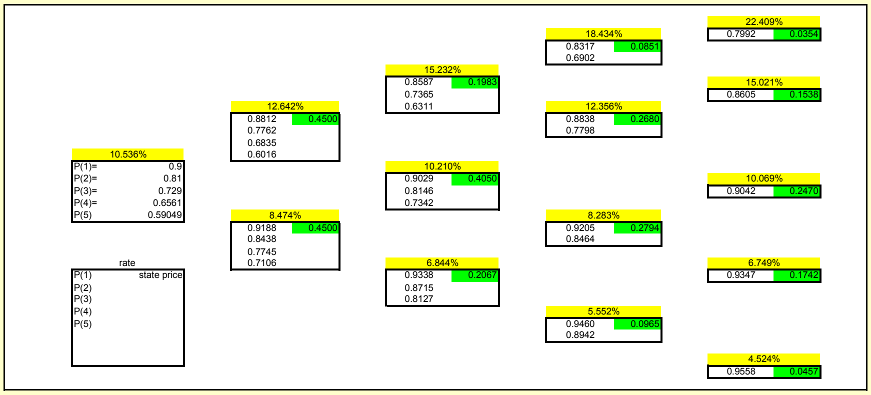
Example

The continuous compounded yield curve is flat at 10.536%   The volatility is 10%



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- The par curve can be computed as flat at 11.11%
- This means that a coupon bond paying 11.11% will be priced at par.

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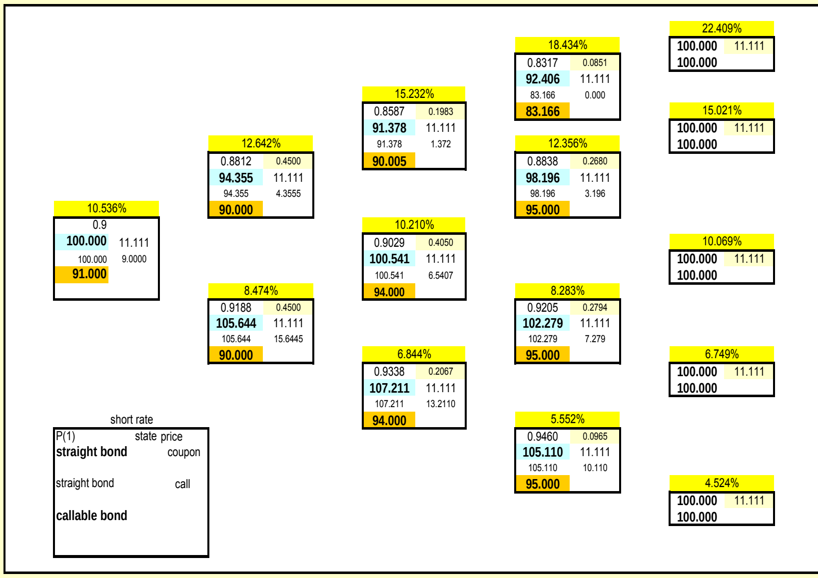
It is!



- Consider a particular node: We compute the bond price at this node, just after a coupon has been paid, as:

$$B = (0.5B^u + 0.5B^d + \text{coupon})d$$

where d is the one period discount factor at the node.

- We can compute the entire term structure at each node, if we want to.
- It may mean that we have to extend the lattice out many periods, but that is why we have computers!

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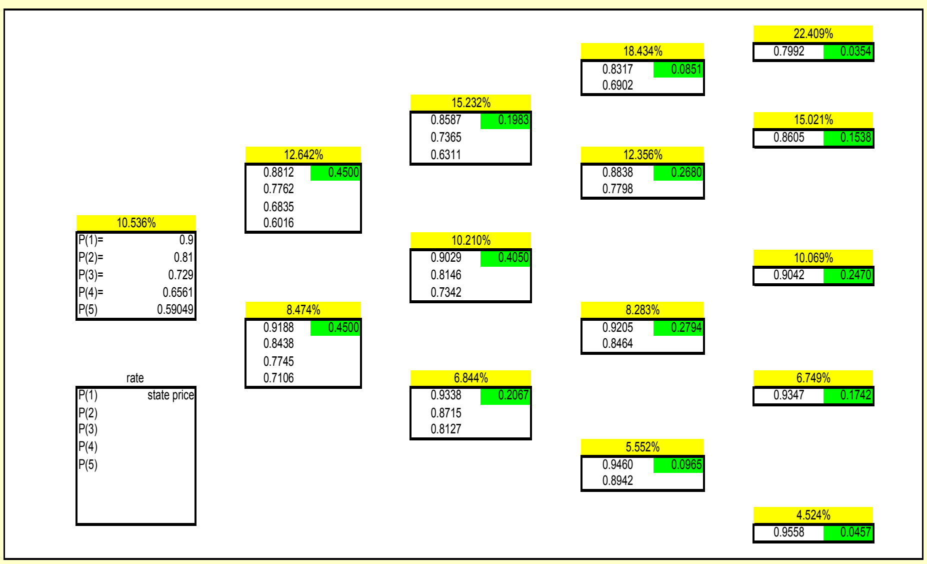

Lets look at the calculations more carefully!

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There are many interest rate models.

- This sequential building procedure is called calibration or external calibration. We could allow sigma to be different at each time period.
- This model is called the Black Derman Toy model.
- There are many possible models that can be established. Clearly there is no unique lattice.
  - For example, we could have fixed the nodes, and played around with the risk neutral probabilities.

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An array of models!

- Cox Ingersoll Ross.
- Vasicek
- Hull and White.
- Heath Jarrow Morton
- Ritchken and Sankarasubramanian
- Black Karazinsky
- Black Derman Toy.
- Market Models…..and more.

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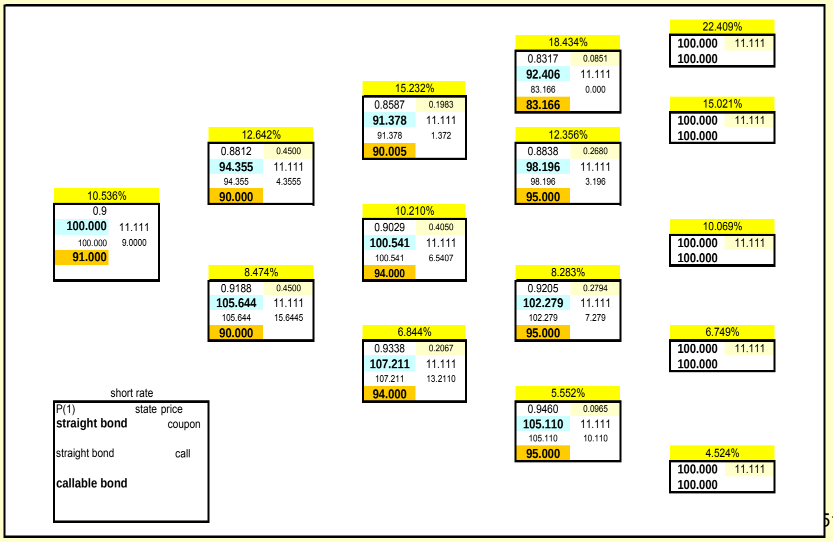
Pricing Options and Callable Bonds
Strike Prices can vary over time -,90, 94, 95, -:



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Pricing a call and a callable bond

- A callable bond is a straight bond – call option on a bond.
- To price the call option, assume we are at a node:
- $C = Max[C^{go}, C^{stop}]$
  - $C^{go} = [0.5C^u + 0.5C^d]d$;
  - $C^{stop} = Max[B – X, 0]$
  - $CB = B – C$

Or

- $CB = Min[B, X] = Min[(0.5B^u+0.5B^d + coupon)d, X]$

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Valuing a Putable Bond

- A bond is likely to be put if the prevailing interest rate is higher than the coupon. Ie the market price of the bond is lower than the par value.
- The process for valuing a putable bond is the same as the process for valuing a callable bond, with one exception
  - The value at each node must be changed to reflect the HIGHER of the value obtained by backward recursion and the put price
- Value of a Putable Bond = Straight Bond +Put Option
- PB = B + P
- P = Max[$(0.5P^u + 0.5P^d)d$, X-B]

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The interest rate lattice

sigma	P(0,1)	P(0,2)	P(0,3)	P(0,4)	P(0,5)	P(0,6)	P(0,7)	P(0,8)
0.2	0.91393	0.83527	0.76338	0.69768	0.63763	0.58275	0.53259	0.48675
yield	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09

								0.3480
							0.2836	
						0.2319		0.2333
					0.1905		0.1901	
				0.1570		0.1555		0.1564
			0.1300		0.1277		0.1274	
		0.1080		0.1053		0.1042		0.1048
rate	0.0900		0.0871		0.0856		0.0854	
		0.0724		0.0706		0.0699		0.0703
			0.0584		0.0574		0.0573	
				0.0473		0.0468		0.0471
					0.0385		0.0384	
						0.0314		0.0316
							0.0257	
								0.0212

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One Period Discount Factors

								0.7061
							0.7531	
						0.7930		0.7919
					0.8266		0.8269	
				0.8547		0.8560		0.8552
			0.8781		0.8801		0.8804	
		0.8977		0.9001		0.9010		0.9005
discount	0.9139		0.9166		0.9180		0.9181	
		0.9302		0.9319		0.9325		0.9321
			0.9433		0.9442		0.9444	
				0.9538		0.9543		0.9540
					0.9623		0.9624	
						0.9691		0.9689
							0.9746	
								0.9791

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8-period discount bond prices over time. Note the initial value matches the input!

									1.00000
								0.70606	
							0.56406		1.00000
						0.49367		0.79191	
					0.46119		0.68101		1.00000
P(0,8)				0.45086		0.62225		0.85523	
			0.45450		0.59385		0.77283		1.00000
		0.46743		0.58428		0.72719		0.90048	
	0.48675		0.58694		0.70441		0.84130		1.00000
		0.59775		0.69644		0.80752		0.93214	
			0.69828		0.79029		0.89059		1.00000
				0.78410		0.86639		0.95399	
					0.85387		0.92526		1.00000
						0.90829		0.96892	
							0.94925		1.00000
								0.97906	
									1.00000

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Date 0 State Prices

								0.00237
							0.00631	
						0.01590		0.02066
					0.03848		0.04422	
				0.09005		0.08858		0.07328
			0.20510		0.16515		0.12495	
		0.45697		0.28145		0.19319		0.13931
state	1.00000		0.41764		0.26255		0.18366	
		0.45697		0.29164		0.20723		0.15465
			0.21253		0.18369		0.14898	
				0.10024		0.10972		0.10090
					0.04780		0.06350	
						0.02300		0.03598
							0.01114	
								0.00543

Pricing cash flows for an exotic contract!

								0.00237
							0.00631	
						0.01590		0.02066
					0.03848		0.04422	
				0.09005		0.08858		0.07328
			0.20510		0.16515		0.12495	
		0.45697		0.28145		0.19319		0.13931
state	1.00000		0.41764		0.26255		0.18366	
		0.45697		0.29164		0.20723		0.15465
			0.21253		0.18369		0.14898	
				0.10024		0.10972		0.10090
					0.04780		0.06350	
						0.02300		0.03598
							0.01114	
								0.00543

								1
							0	
						1		1
					0		0	
				0		1		1
			0		0		0	
cash flows		0		0		1		1
	0		0		0		0	
		0		0		1		1
			0		0		0	
				0		1		1
					0		0	
						1		1
							0	
								1

								0.00237
							0.00000	
						0.01590		0.02066
					0.00000		0.00000	
				0.00000		0.08858		0.07328
date 0 state values			0.00000		0.00000		0.00000	
		0.00000		0.00000		0.19319		0.13931
	0.00000		0.00000		0.00000		0.00000	
		0.00000		0.00000		0.20723		0.15465
			0.00000		0.00000		0.00000	
				0.00000		0.10972		0.10090
					0.00000		0.00000	
						0.02300		0.03598
							0.00000	
								0.00543

Portfolio Value
1.1702

Choose a coupon bond with coupons at the par rate

P(0,1)	P(0,2)	P(0,3)	P(0,4)	P(0,5)	P(0,6)	P(0,7)	P(0,8)
0.913931	0.83527	0.763379	0.697676	0.637628	0.582748	0.532592	0.486752
0.094174	0.094174	0.094174	0.094174	0.094174	0.094174	0.094174	0.094174

coupon	*9.417428*
face	*100*

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Pricing a coupon bond

coupon bond

								100.000
							77.256	
						68.810		100.000
					67.383		86.649	
				69.947		82.301		100.000
			75.091		83.028		93.577	
		82.078		86.934		92.852		100.000
	90.477		93.012		95.683		98.528	
100.000		100.669		100.902		100.699		100.000
	109.523		107.816		105.317		101.993	
		115.982		111.659		106.340		100.000
			119.261		112.352		104.383	
				119.580		110.302		100.000
					117.348		106.017	
						113.043		100.000
							107.126	
								100.000

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Pricing a call option on a coupon bond

call strikes	130	120	115	110	105	102	100	100	100
									0.000
								0.000	
							0.000		0.000
						0.000		0.000	
					0.000		0.000		0.000
				0.078		0.000		0.000	
call			0.414		0.181		0.000		0.000
option		1.169		0.865		0.412		0.000	
on coupon	2.407		2.190		1.742		0.915		0.000
bond		4.099		3.914		3.382		1.993	
			6.623		6.659		6.340		0.000
				10.129		10.352		4.383	
					14.580		10.302		0.000
						15.348		6.017	
							13.043		0.000
								7.126	
									0.000

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Pricing a callable bond

call strikes	130	120	115	110	105	102	100	100	100
									100.000
								77.256	
							68.810		100.000
						67.383		86.649	
callable					69.947		82.301		100.000
bond				75.013		83.028		93.577	
			81.664		86.753		92.852		100.000
		89.308		92.146		95.271		98.528	
	97.593		98.478		99.160		99.785		100.000
		105.424		103.902		101.934		100.000	
			109.359		105.000		100.000		100.000
				109.132		102.000		100.000	
					105.000		100.000		100.000
						102.000		100.000	
							100.000		100.000
								100.000	
									100.000

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Pricing a putable bond

strike	90	92	94	96	100	100	100	100	100
put									
									0.000
								22.744	
							31.190		0.000
						32.617		13.351	
					30.053		17.699		0.000
				20.909		16.972		6.423	
			12.182		13.066		7.148		0.000
put		7.092		6.837		4.317		1.472	
option	4.123		3.619		2.126		0.676		0.000
		1.931		1.060		0.315		0.000	
			0.533		0.149		0.000		0.000
				0.071		0.000		0.000	
					0.000		0.000		0.000
						0.000		0.000	
							0.000		0.000
								0.000	
									0.000

strike	90	92	94	100	100	100	100	100	100
									100.000
								100.000	
							100.000		100.000
						100.000		100.000	
					100.000		100.000		100.000
				96.000		100.000		100.000	
			94.261		100.000		100.000		100.000
putable		97.569		99.848		100.000		100.000	
bond	104.123		104.288		103.028		101.375		100.000
		111.454		108.876		105.632		101.993	
			116.515		111.808		106.340		100.000
				119.332		112.352		104.383	
					119.580		110.302		100.000
						117.348		106.017	
							113.043		100.000
								107.126	
									100.000

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Pricing a callable/putable bond

- Price of a callable/putable bond = price of a straight bond – price of a call option+ price of a put option.
- Is this correct?

coupon - call +put	101.716

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Pricing a callable putable bond

calll strike	130	120	115	110	105	102	100	100	100
put strike	90	92	94	96	98	100	100	100	100

									100.000
								100.000	
							100.000		100.000
						100.000		86.649	
					98.000		100.000		100.000
				96.000		100.000		93.577	
			94.000		98.000		100.000		100.000
callable/		96.300		98.198		100.000		98.528	
putable	101.580		101.722		101.361		100.000		100.000
bond		107.158		104.927		102.000		100.000	
			109.842		105.000		100.000		100.000
				109.132		102.000		100.000	
					105.000		100.000		100.000
						102.000		100.000	
							100.000		100.000
								100.000	
									100.000

callable/putable bond	101.580
coupon - call +put	101.716

An exotic Interest Rate Swap with a Cap and with payments NOT paid in arrears, but paid up-front. The swap is for seven years

								0.706064
							0.753097	0.34805
						0.792995	0.283562	0.791912
				0	0.826574	0.231938	0.826895	0.233305
				0.854683	0.190466	0.856011	0.190077	0.855227
			0.878135	0.157025	0.880141	0.155473	0.88037	0.156389
		0.897669	0.129955	0.900093	0.127673	0.90103	0.127412	0.900477
discount	0.913931	0.107953	0.916575	0.105257	0.917978	0.104216	0.918138	0.104831
rate	0.09	0.930193	0.087111	0.931876	0.085582	0.932526	0.085407	0.932142
		0.072363	0.94328	0.070556	0.944247	0.069858	0.944358	0.07027
			0.058392	0.953806	0.057367	0.954252	0.05725	0.953989
				0.047295	0.962276	0.046827	0.962351	0.047103
					0.038454	0.969098	0.038376	0.968919
						0.031389	0.974604	0.031574
							0.025724	0.979057
								0.021165

cap	0.12
not in arrears	

								0.706064
							0.753097	0.12
						0.792995	0.12	0.791912
					0.826574	0.12	0.826895	0.12
				0.854683	0.12	0.856011	0.12	0.855227
			0.878135	0.12	0.880141	0.12	0.88037	0.12
		0.897669	0.12	0.900093	0.12	0.90103	0.12	0.900477
discount	0.913931	0.107953	0.916575	0.105257	0.917978	0.104216	0.918138	0.104831
payment	0.09	0.930193	0.087111	0.931876	0.085582	0.932526	0.085407	0.932142
		0.072363	0.94328	0.070556	0.944247	0.069858	0.944358	0.07027
			0.058392	0.953806	0.057367	0.954252	0.05725	0.953989
				0.047295	0.962276	0.046827	0.962351	0.047103
					0.038454	0.969098	0.038376	0.968919
						0.031389	0.974604	0.031574
							0.025724	0.979057
								0.021165

Vanilla Swap vs an exotic swap!

								0.002375
							0.006306	12
						0.015905	12	0.020657
					0.038484	12	0.04422	12
				0.090054	12	0.088582	12	0.073282
			0.205102	12	0.16515	12	0.124946	12
		0.456966	12	0.281451	12	0.193185	12	0.139311
State Price		10.79534	0.417635	10.5257	0.26255	10.42164	0.183657	10.48305
pay	0	0.456966	8.711113	0.291636	8.558189	0.207231	8.540713	0.154655
		7.23633	0.212533	7.055588	0.183689	6.985833	0.148977	7.027002
			5.839234	0.100239	5.736725	0.109724	5.725011	0.100897
				4.729502	0.047804	4.682744	0.063497	4.71034
					3.845442	0.023	3.83759	0.035984
						3.138937	0.011145	3.157435
							2.572413	0.005431
								2.116492
	0	8.23985	6.863021	6.574855	6.978679	5.30084	5.912539	4.303303

Value of Floating	=	44.17309
Annuity Factor	=	4.963225
Swap Rate	=	8.900077
Vanilla Swap	=	9.417428

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Futures Pricing

Lattice for futures prices. The underlying bond at delivery in period 5, is a 3 period discount bond. FU(5,8)

									1
								0.706064	
							0.56406		1
						0.493666		0.791912	
					0.461194	0.493666	0.681006		1
P(0,8)				0.450864	0.557959	0.622251		0.855227	
			0.454498	0.61634	0.59385	0.622251	0.772835		1
		0.467432	0.668689	0.584279	0.674722	0.727192		0.900477	
	0.486752	0.715142	0.586937	0.721038	0.704414	0.727192	0.841299		1
	0.756007	0.597751	0.761595	0.696439	0.767354	0.807516		0.932142	
		0.796873	0.698282	0.802153	0.79029	0.807516	0.890591		1
			0.83215	0.784103	0.836952	0.866388		0.953989	
				0.862146	0.853865	0.866388	0.925256		1
					0.88734	0.908292		0.968919	
P(0,8)	0.486752					0.908292	0.949253		1
P(0,5)	0.637628							0.979057	
FO(5,8)=	0.7634								1
FU(5,8)=	0.7560								
% diff	0.975139								

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Pricing of a Futures Contract

- Pay 0 today:
- Tomorrow two things can happen:
- Profit = $F^{up} – F$ or
- Profit = $F^{down} – F$

- Hence Expected profit is

  $0.5(F^{up} – F) + 0.5(F^{down} – F) = 0.5(F^{up} + F^{down}) – F$

  What should this number be equal to?

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Pricing of a Futures Contract

- Pay 0 today:
- Tomorrow two things can happen:
- Profit = $F^{up} – F$ or
- Profit = $F^{down} – F$
- Hence Expected profit is:

  $0.5(F^{up} – F) + 0.5(F^{down} – F) = 0.5(F^{up} + F^{down}) – F = 0$

  Hence $F = 0.5(F^{up} + F^{down})$

  That is F is just the average of the next F values. The last F values are known, so we can use backward recursion!

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Options on Futures

strike	0.75
expiration	4

Lattice for futures prices. We now consider pricing American options on the futures. The expiration date of the option is period 4. The futures settles in Period 5. The underlying discount bond matures in period 8.

A-call

					0.493666
				0.557959	
			0.61634	0	0.622251
		0.668689	0	0.674722	
	0.715142	0.003429	0.721038	0	0.727192
0.756007	0.013873	0.761595	0.00781	0.767354	
0.02964	0.796873	0.02748	0.802153	0.017354	0.807516
	0.050989	0.83215	0.052153	0.836952	
		0.08215	0.862146	0.086952	0.866388
			0.112146	0.88734	
				0.13734	0.908292

E-call	0.027094
A-call	0.02964

A-put

					0.493666
				0.557959	
			0.61634	0.192041	0.622251
		0.668689	0.13366	0.674722	
	0.715142	0.081311	0.721038	0.075278	0.727192
0.756007	0.043464	0.761595	0.033879	0.767354	
0.023161	0.796873	0.015526	0.802153	0	0.807516
	0.007221	0.83215	0	0.836952	
		0	0.862146	0	0.866388
			0	0.88734	
				0	0.908292

A-put	0.023161
E-Put	0.019823

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Options on Forwards

P(0,1)	P(0,2)	P(0,3)	P(0,4)	P(0,5)	P(0,6)	P(0,7)
0.913931	0.83527	0.763379	0.697676	0.637628	0.582748	0.532592

Forwards

P(0,8)
P(0,5)
Forward
call
put

					0.493666
				0.461194	
				0.826574	
				0.557959	
			0.450864	0	
			0.72935	0.192041	
			0.618173		0.622251
		0.454498	0	0.59385	
		0.675544	0.114237	0.880141	
		0.672788		0.674722	
	0.467432	0.003429	0.584279	0	
	0.648142	0.065033	0.809237	0.075278	
	0.721188		0.722012		0.727192
0.486752	0.013142	0.586937	0.00781	0.704414	
0.637628	0.036158	0.768511	0.033879	0.917978	
0.763379		0.763733		0.767354	
0.027094	0.597751	0.025852	0.696439	0.017354	
0.019823	0.747211	0.015526	0.867681	0	
	0.799977		0.802644		0.807516
	0.046148	0.698282	0.0486	0.79029	
	0.007221	0.838061	0	0.944247	
		0.833212		0.836952	
		0.073371	0.784103	0.086952	
		0	0.909226	0	
			0.862384		0.866388
			0.106966	0.853865	
			0	0.962276	
				0.88734	
				0.13734	
				0	
					0.908292

A-call	0.027094
A-put	0.019823

E-call	0.027094
E-put	0.019823

Options on Forwards are never exercised early. Options on Futures may be exercised early.

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Options on Forwards

- If you exercise early, you get a forward contract at a strike of X.
  - The "market" gets forward contracts at a price of FO say.
- Therefore the value of exercising is the present value of the difference!
- Early exercise of Options on Forwards is never optimal.

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Comparing Futures with Forwards on zero coupon bonds

Comparing Futures Prices with Forward Prices.

						0.493666
					0.557959	0.493666
				0.61634	0.557959	0.622251
			0.668689	0.618173	0.674722	0.622251
		0.715142	0.672788	0.721038	0.674722	0.727192
fu	0.756007	0.721188	0.761595	0.722012	0.767354	0.727192
fo	0.763379	0.796873	0.763733	0.802153	0.767354	0.807516
		0.799977	0.83215	0.802644	0.836952	0.807516
			0.833212	0.862146	0.836952	0.866388
				0.862384	0.88734	0.866388
					0.88734	0.908292
						0.908292

The underlying is an 8 year zero coupon bond.
The contract requires delivery in 5 years.

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Corporate Bonds

- Sinking Funds and Callable Bonds
- Putable Bonds, Floaters with Caps and Floors etc.
- Innovations in Sinking Funds…eg The Double up option.
- Embedded options in futures contracts.

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Option Adjusted Spreads

- The cheapness or expensiveness of any two non Treasury securities can be determined by comparing their yield spreads over the benchmark interest rate (say US treasury spot rates)
- The option adjusted spread is the constant spread that must be added to all one period forward rates on the binomial tree that will make the arbitrage free (theoretical) value of an option embedded bond equal to its market price.

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Option Adjusted Spreads

- Are determined through a “trial and error process.”
  - If the market price is lower than the theoretical price, then you increase the OAS
  - If the market price is higher than the theoretical price, then you decrease the OAS.
- The spread is known as “option adjusted” because the cash flows are adjusted for the impact of interest rate volatility (and other issues) on the embedded option.

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Option Adjusted Spreads

- Say you had to add 50 basis points at each node in the lattice so that our callable bond equated with the market price. Then the option adjusted spread would be 50 basis points.
- The OAS measures credit risk and liquidity risk if the benchmark interest rate is the US Treasury spot rate. If the benchmark is the issuers spot rate curve, then OAS measures only the liquidity risk of the bond since the credit risk is already factored into the issuers spot rate curve.

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Effective Duration and Convexity using the Binomial tree

- Duration is the approximate percentage change for a 100 basis change in all interest rates.

$$Duration = (V_{-} - V_{+}) / 2V(\Delta y)$$

$$Convexity = [V_{+} - 2V + V_{-}] / 2V(\Delta y)^2$$

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Computational methodology.

- Calculate the OAS
- Impose a small parallel upward shift on the yield curve +OAS.
- Build a new interest rate tree using the new curve.
- Compute the value V(+) using the modified tree.
- Repeat the steps using a downward shift. Obtain the value V(-)
- Plug the values V(+) V, V(-) into the duration and convexity formulae.

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- Say for our corporate callable bond the OAS = 50 basis points. After computing the perturbed values and plugging the numbers into the equations we get
  - Duration = 3.26
  - Convexity = -60.145
- How do we interpret this?

Summary

- Internally Consistent Prices
- Externally Consistent Prices
- The Black Derman Toy Model
  - Callable Bonds
  - Putable Bonds
  - Callable Putable Bonds
  - Exotic Swaps with embedded Options
  - Futures and Forwards
  - Options on Futures and Options on Forwards

Summary

- Often compute the Option Adjusted Spread.
  - Used for Corporate Bonds, Mortgage backed Securities and others.
- Can compute the Effective Duration and Convexity.
- Can also compute Key rate Durations.
- There are many different models of the Term Structure of Interest Rates.
- Calibration Issues…..

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