Pricing Futures and Forwards

Peter Ritchken

Objectives

- You will learn
  - how to price a forward contract
  - how to price a futures contract
  - the relationship between futures and forward prices
  - the relationship between futures prices and expected prices in the future.
- You will use
  - arbitrage relationships
  - become familiar with the cost of carry model
  - learn how to identify mispriced contracts.
Forward Curves

- Forward Prices are linked to Current Spot prices.
  - The forward price for immediate delivery is the spot price.
  - Clearly, the forward price for delivery tomorrow should be close to today’s spot price.
  - The forward price for delivery in a year may be further disconnected from the current spot price.
  - The forward price for delivery in 5 years may be even further removed from the current spot price.

Forward Prices of West Texas Intermediate Crude Oil.

- A Contango Market
Forward Prices of West Texas Intermediate Crude Oil.

- A Backwardation Market

Forward Prices

Time to Expiration

Peter Ritchken Forwards and Futures Prices

Forward Prices of West Texas Intermediate Crude Oil.

- Short term Backwardation/Long term Contango

Forward Prices

Time to Expiration

Peter Ritchken Forwards and Futures Prices
Forward Prices of West Texas Intermediate Crude Oil.

- Mixed Contango/Backwardation Forward Curve

Forward Prices

Time to Expiration

Forward Prices of Heating Oil.

- Peaks in Winter and lows in Summer.

Time to Expiration
Forward Prices of Electricity.

- Peaks in Winter and Summer with lows in winter and fall.

What determines the term structure of forward prices?

- How can we establish the fair forward price curve?
- Does the forward curve provide a window into the future?
  - Do forward prices predict future expected spot prices?
  - What can we learn from forward prices?
- Do futures prices equal forward prices?
- What can we learn from futures prices?
Forward and Futures Prices

- We make the following assumptions:
  - No delivery options.
  - Interest rates are constant.
  - This means there is only one grade to be delivered at one location at one date.
  - $S(0)$ is the underlying price. $F(0)$ is the forward price and $T$ is the date for delivery.

The Value of a Forward Contract

- At date 0: $V(0) = 0$
- At date $T$: $V(T) = S(T) - F(0)$
- What about $V(t)$?
Determining $V(t)$

<table>
<thead>
<tr>
<th></th>
<th>Value at 0</th>
<th>Value at $t$</th>
<th>Value at $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy Forward at date 0</td>
<td>0</td>
<td>$V(t)$</td>
<td>$S(T)-F(0)$</td>
</tr>
<tr>
<td>Sell a forward at date $t$</td>
<td>-</td>
<td>0</td>
<td>$-(S(T)-F(t))$</td>
</tr>
<tr>
<td>Value of Strategy</td>
<td>$V(t)$</td>
<td></td>
<td>$F(t) - F(0)$</td>
</tr>
</tbody>
</table>

What is $V(t)$?

- $V(t) = \text{Present Value of } F(t) - F(0)$.
- BUT $F(t)$ and $F(0)$ are known at date $t$.
- Hence the payout is certain.
- Hence we have:

$$V(t) = \exp(-r(T-t))[F(t)-F(0)]$$
Property

- The value of a forward contract at date \( t \), is the change in its price, discounted by the time remaining to the settlement date.

- Futures contracts are marked to market. The value of a futures contract after being marked to market is zero.

Property

- If interest rates are certain, forward prices equal futures prices.

- Is this result surprising to you?
### With One Day to Go

<table>
<thead>
<tr>
<th></th>
<th>Initial Value</th>
<th>Final Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long 1 forward</td>
<td>0</td>
<td>( S(T) - FO(T-1) )</td>
</tr>
<tr>
<td>Short 1 futures</td>
<td>0</td>
<td>(- (S(T) - FU(T-1)))</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>( FU(T-1) - FO(T-1) )</td>
</tr>
</tbody>
</table>

### With Two Days to Go

<table>
<thead>
<tr>
<th></th>
<th>Initial Value</th>
<th>Final Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long 1 forward</td>
<td>0</td>
<td>([FO(T-1) - FO(T-2)]B(T-1,T))</td>
</tr>
<tr>
<td>Short B(T-1,T) futures</td>
<td>0</td>
<td>(-[FU(T-1) - FU(T-2)]B(T-1,T))</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>([FU(T-2)-FO(T-2)]B(T-1,T))</td>
</tr>
</tbody>
</table>
### With Three Days to Go

<table>
<thead>
<tr>
<th></th>
<th>Initial Value</th>
<th>Final Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Long 1 forward</strong></td>
<td>0</td>
<td>[FO(T-2) - FO(T-3)]B(T-2,T)</td>
</tr>
<tr>
<td><strong>Short B(T-1,T) futures</strong></td>
<td>0</td>
<td>-[FU(T-2) - FU(T-3)]B(T-2,T)</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>[FU(T-3)-FO(T-3)]B(T-2,T)</td>
</tr>
</tbody>
</table>

### Example:
**Tailing the Hedge**

- Little Genius sells a Forward Contract. Then hedges this exposure by taking a long position in $x$ otherwise identical futures contracts.
  - What should $x$ be?
- With $T$ years to go to expiration, the number of futures contracts to purchase is $x = \exp(-rT)$
- The strategy is dynamic, since the number of futures to hold changes over time. (Actually increases to 1 as $T$ goes to 0)
Example

- $FO(0) = FU(0) = F(0) = 100$
- $F(1) = 120; F(2) = 150; F(3) = 160$

- **Profit on Forward:**
  
  $160 - 100 = 60$

- **Profit on Futures:**
  
  $20 \exp(r \times \frac{2}{365}) + 30 \exp(r \times \frac{1}{365}) + 10.$

Example: Now Tail the Hedge.

- With 3 days to go: Buy $N_3 = \exp(-r \times \frac{2}{365})$ futures.
- With 2 days to go: Buy $N_2 = \exp(-r \times \frac{1}{365})$ futures.
- With 1 day to go: Buy $N_1 = 1$ futures.

- **Profit on this strategy is**
  
  $20N_3 e^{r2/365} + 30N_2 e^{r1/365} + 10 = 20 + 30 + 10 = 60$

- This is the payout of a forward.
Property

- If futures prices are positively correlated with interest rates, then futures prices will exceed forward prices.
- If futures prices are negatively correlated with interest rates, then futures prices will be lower than forward prices.

“Proof” For Positive Correlation.

- FU prices increase, the long wins and invests the proceeds at a high interest rate.
- FU prices decrease, the long loses, but finances the losses at a lower interest rate.
- Overall, the long in the futures contract has an advantage.
- The short will not like this, and will demand compensation in the form of a higher price.
Pricing of Forward Contracts

- Consider an investment asset that provides no income and has no storage costs. (Gold)
- If the forward price, relative to the spot price, got very high, perhaps you would consider buying the gold and selling forward.
- If the forward price, relative to the spot price, got very low, perhaps you would consider buying the forward, and selling the asset short!

- Let's take a closer look at the restriction these trading schemes impose on fair prices.

Pricing Forward Contracts

- Little Genius starts off with no funds. If they buy an asset, they must do so with borrowed money. We first consider the following strategy:
  - Buy Gold, by borrowing funds. Sell a forward contract.
  - At date T, deliver the gold for the forward price. Pay back the loan.

- Profit = F(0) - S(0)exp(rT)
Cost of Carry Model

- Clearly if \( F(0) > S(0) \exp(rT) \), then Little Genius would do this strategy. Starting with nothing they lock into a profit of \( F(0) - S(0) \exp(rT) > 0 \).
- To avoid such riskless arbitrage, the highest the forward price could go to is \( S(0) \exp(rT) \).
- \( F(0) < S(0) \exp(rT) \).

Reverse Cash and Carry:
(In a Perfect Market)

- Note that profit from the strategy is known at date 0!
- If positive, Little Genius does the strategy!
- If negative, Little Genius does the opposite!
- That is LG buys the forward contract, and sells gold short. Selling short generates income which is put into riskless assets.
- Profit = \( S(0) \exp(rT) - F(0) \)
Reverse Cash and Carry: (In a Perfect Market)

- If Profit = \( S(0) \exp(rT) - F(0) > 0 \), Little Genius would make riskless arbitrage profits.
- Hence:
  - \( S(0) \exp(rT) < F(0) \)
  - That is, to avoid riskless arbitrage, the forward price must be bigger than the future value of a riskless loan of \( S(0) \) dollars.
- Hence: \( S(0) \exp(rT) < F(0) < S(0) \exp(rT) \),
- Or \( F(0) = S(0) \exp(rT) \)

Term Structure of Gold Futures Prices (In a Perfect Market)

- A Contango market for Gold!

\[
F = S(0) \exp(rT)
\]

Pricing Futures and Forwards by Peter Ritchken
Arbitrage Restriction

To avoid riskless arbitrage, the price of a forward contract on Gold is:

\[ F(0) = S(0)e^{rT} \]

Example:

- \( S(0) = 400; F(0) = 450; T = 1 \) year;
- Simple Interest Rate = 10% per year.

- Borrow $400 for 1 year at 10% \(+400\)
- Buy 1 ounce of Gold \(-400\)
- Sell 1 forward contract \(0\)
- Net cash flow \(0\)
Example
(continued)

- After 1 year:
  - Remove gold from storage and deliver: 450
  - Repay loan, including interest \(-440\)

  \[
  \text{Net Cash Flow} = \quad +$10
  \]

- Initial Investment = $0.
- To avoid arbitrage free profits from this strategy:
  \[ F(0) < 440 \]

Example
Cash and Carry with Market Imperfections

- \(S(0) = 400; F(0) = 450;\)
  - Simple interest rate = 10%;
  - Transaction Cost = 3\% of spot.

- At Date 0:
  - Sell Forward Contract
  - Borrow $412 at 10%..........+412
  - Buy 1oz. Of Gold.................-412
  - Net Cash Outflow............... 0
Cash and Carry with Market Imperfections (continued)

• At Date T:

Remove Gold from Storage and
Deliver .................................... +450.00
Repay Loan............................ - 453.20
Total Cash Flow...................... -3.20

• Hence,
    \[ F(0) < 453.20 \]

Reverse Cash and Carry with Market Imperfections

• \( S(0) = 400; F(0)=450; \) Interest Rate = 10%
  Transaction Cost = 3%

• At Date 0

Sell 1oz. gold short
Receive 400(0.97) ......................... $388
Invest proceeds at riskless rate......-$388
Net Cash Flow.......................... $0
Reverse Cash and Carry with Market Imperfections
(continued)

- At Date T:

Accept loan proceeds
[388(1.1)].......................426.80
Accept gold delivery....-450.00
Total Cash Flow..........-23.20

- Therefore, you would have arbitrage free profits if
  \[ F(0) < 426.80 \]

Arbitrage Free Bounds

- Hence, to avoid riskless arbitrage:
  \[ 426.8 < F(0) < 453.2 \]

- The size of the bounds increase with market imperfections.
- However, the actual size of the bounds are determined by the market players that face the least imperfections!
Pricing Futures Contracts on Stocks

- $S(0) = 100$
- $d = $2 in 0.5 years.
- Interest = 10% simple.
- Forward contract is for 1 year

- At Date 0
  - Forward contract for 1 year.
  - Borrow $100 ..........+100
  - and Buy stock.........-100

At Date $t=0.5$
- Receive $2.
- Invest $2 at 10% per year

At date $T=1$
- Collect proceeds from dividend.......$2.10
- Sell stock for forward price...............$F(0)$
- Repay loan.................................-$110$
- Profit = $F(0)-110+2.10$
- Profit is positive if $F(0)$ exceeds 108.90


Futures Prices on Stock

To avoid arbitrage opportunities the forward price is bounded by:

\[ F(0) \leq S(0)e^{rT} - de^{r(T-t)} \]
\[ F(0) \leq S(0)[1 + rT] - d[1 + r(T-t)] \]

Forward Prices on a Stock

- To avoid arbitrage opportunities the forward price of a stock is
  - \( F(0) = S(0)e^{rT} - d e^{r(T-t)} \)
  - \( F(0) = S(0)[1 + rT] - d[1 + r(T-t)] \)
Futures Price on a Nondividend Paying Stock

- Does the futures curve provide any forecasting power for the future stock price?
- If the slope of the futures curve increases
  - The prospects for the stock has improved?
  - Interest Rates have increased?

Stock Market Indices

<table>
<thead>
<tr>
<th></th>
<th>Time 0</th>
<th>Time t</th>
<th>Shares Outstanding</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>150</td>
<td>150</td>
<td>50</td>
</tr>
<tr>
<td>B</td>
<td>40</td>
<td>80</td>
<td>100</td>
</tr>
<tr>
<td>C</td>
<td>10</td>
<td>30</td>
<td>500</td>
</tr>
</tbody>
</table>
### Market Value Indices

- $MV(0) = 150(50) + 40(100) + 10(500) = 16500$
- $MV(t) = 150(50) + 80(100) + 30(500) = 30500$

- $I(0) = 100$
- $I(t) = I(0)[MV(t)/MV(0)]$
  \[= 100[30500/16500] = 184.85\]

### Price Weighted Index

- $V(0) = [150 + 40 + 10]/3 = 66.667$
- $V(t) = [150 + 80 + 30]/3 = 86.67$

- $I(t) = [V(t)/V(0)]I(0)$
  \[= [86.67/66.667]100\]
  \[= 130\]
Pricing Futures on Price Weighted Indices

- Stocks in the index are A and B.
- A pays a dividend of size $d_A$ at time $t_A$.
- B pays a dividend of size $d_B$ at time $t_B$.

\[
F(0) = S_A(0)e^{rT} - d_Ae^{r(T-t_A)} + S_B(0)e^{rT} - d_Be^{r(T-t_B)}
\]

\[
F(0) = [S_A(0) + S_B(0)]e^{rT} - [d_Ae^{r(T-t_A)} + d_Be^{r(T-t_B)}]
\]

\[
F(0) = I(0)e^{rT} - FV \quad (\text{dividends})
\]

Stock Index Arbitrage with a Value Weighted Index

<table>
<thead>
<tr>
<th>Price</th>
<th>Shares</th>
<th>Div</th>
<th>Time to Div.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 40</td>
<td>1m</td>
<td>0.5</td>
<td>10 days</td>
</tr>
<tr>
<td>B 35</td>
<td>2m</td>
<td>0.5</td>
<td>12 days</td>
</tr>
<tr>
<td>C 25</td>
<td>2m</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Pricing Futures and Forwards by Peter Ritchken

Peter Ritchken

Forwards and Futures Prices

47

48
Example:
Stock Index Arbitrage with Price Weighted Index

- Composition of Index:
  - \( A = \frac{40}{40+35+25} = \frac{1}{4} \)
  - \( B = \frac{7}{16} \)
  - \( C = \frac{5}{16} \)
  - \( I(0) = 400 \)
  - Multiplier = 500
  - Each contract controls \( 400(500) = $200,000 \).

---

Stock Index Arbitrage

- Portfolio Composition:
  - \( \frac{1}{4} \times 200,000 = $50,000 \) or 1,250 shares of A
  - \( \frac{7}{16} \times 200,000 = $87,500 \) or 2,500 shares of B
  - \( \frac{5}{16} \times 200,000 = $62,500 \) or 2,500 shares of C
### Stock Index Arbitrage

<table>
<thead>
<tr>
<th>Stock</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Borrowed Funds</td>
<td>50,000</td>
<td>87,500</td>
<td>62,500</td>
</tr>
<tr>
<td>Number Shares</td>
<td>1,250</td>
<td>2,500</td>
<td>2,500</td>
</tr>
<tr>
<td>Dividend Income Received and Invested at 10%</td>
<td>628.43</td>
<td>1256.18</td>
<td>-</td>
</tr>
</tbody>
</table>

Amount owed = 200,000exp(0.10(30/365))

\[
= 201,650.61
\]

Less dividends and interest.............. -$1,844.61.

Total = ............................................... $199,766

Theoretical Futures Price =

\[
= \frac{199,766}{500}
= 399.53
\]

Note, in this example I(0) = 400

---
Stock Index Futures Prices and Index Prices

- If Stock index futures prices are upward sloping as a function of maturity, then market participants are forecasting that the market will increase (i.e., the sentiment is bullish).

- Is this true?

Forward Contracts on FOREX

- Price of 1 British Pound is \( S(0) \) dollars.
- Strategy 1:
  - Invest \( S(0) \) dollars at the risk free rate to obtain \( S(0) \exp(r_D T) \) dollars at date \( T \).
  - \( S(0) \) dollars grows to \( S(0) \exp(r_D T) \) dollars.
Forward Contracts on FOREX

- Strategy 2:
  - Sell $e^{r_F T}$ forward contracts on the pound. Each contract gives the obligation to exchange 1 pound for $F(0)$ dollars.
  - Buy 1 British pound. Invest in riskless rate in Britain to obtain $e^{r_F T}$ pounds.
  - Deliver the pounds for $e^{r_F T} F(0)$ dollars.
  - $S(0)$ grows to $e^{r_F T} F(0)$

- To avoid arbitrage opportunities:
  - $S(0)e^{r_D T} = e^{r_F T} F(0)$
  - or
  
  $F(0) = S(0)e^{(r_F-r_D)T}$

- Forward prices relate to spot prices depending on interest rate differentials.
Pricing Futures Contracts on Dividend Paying “Stocks”

- Assume interest rate is $r$ and storage costs are a fixed percent of the spot price. The underlying pays no dividends.
- $F(0) = S(0)\exp[(r+u)T]$
- Now assume the underlying pays a continuous dividend yield, expressed as a percent of the price. Then
- $F(0) = S(0)\exp[(r+u-d)T]$

The spot price of the S&P500 index is 1000. The dividend yield is 3% per year. Interest rates are 5% continuously compounded. Storage costs are 0%. A one year futures contract should have a futures price of
- Then $F(0) = S(0)e^{(r-d)T} = 1000e^{0.02} = 1020.20$
Forward Prices of a Property

- You can buy a property now for $100m. Alternatively, you can enter into a forward contract to purchase the property in two years time.
  - The property has a maintenance fee that is 1% of the price. The property has buildings that provide rental income that is estimated at 6% of the price per year.
- The fair forward price of the property is
  \[100e^{(r+u-d)T} = 100e^{(0.05+0.01-0.06)2} = 100\text{ million dollars}].\]

Futures Prices of Storable Commodities.

- Commodity forward contracts have two important features that are not present when the underlying is a financial asset.
  - Storage Costs
  - Convenience Yields
- Storage Costs
  - Warehouse space, transportation costs, spoilage, insurance.
  - We will represent these charges as a fraction of the market price of the commodity.
  - If storage costs are 20%, this implies that the annualized storage costs are about 20% of the spot market price.
Futures Prices of Storable Commodities.

- **The convenience yield**
  - Unlike securities, commodities are usually consumed or used in a production process.
  - Having a commodity on hand has value since it allows the production process to continue without disruptions.
  - If the commodity is abundant then there is not much benefit from having it in storage. However,
  - If the commodity is scarce, then having the commodity in inventory is very beneficial.
  - One can view the potential benefit of having a commodity on hand as a yield, just like a continuous dividend yield. We express the convenience yield as a percent of the price in an annualized form.

Futures Prices of Storable Commodities.

- A 2% convenience yield means that having the item readily available comes at a cost that accrues at a rate of 2% of the current price of the commodity per year.

Example:
Spot price = 100; Interest Rate = 5%; Storage Cost = 8%
Convenience Yield = 7%. Time to expiration is 1 year.

\[ F(0) = S(0) e^{(r+u-k)T} = 100 e^{(0.05+0.08 -0.07)} = 106.18 \]
Upper Bound for Commodity Forwards: The Cost of Carry Model

- If forward price is very high:
  - Sell the forward contract
  - Buy the commodity using borrowed funds.
  - Pay the storage fees with borrowed funds
  - Deliver the commodity for the forward price.
- Profit is:
  \[ F(0) - S(0)e^{(r+u)T} \]
  Clearly, to avoid riskless arbitrage:
  \[ F(0) < S(0)e^{(r+u)T} \]

Lower Bound for Commodity Forwards

- If the forward price was very low we would like to initiate the reverse cost of carry. In particular, we need to short sell the commodity.
- We must borrow the commodity from party, A, and then sell it.
- This deprives A from a convenience yield. A is giving up having inventory around to meet unexpected needs! Clearly A needs to be compensated for this.
- This compensation is the convenience yield, represented by \( k\% \) per annum as a fraction of the commodity price
Lower Bound for Commodity Forwards

- A does get to save the storage fees, represented by u% per annum as a fraction of the commodity price.
- So A requires a net compensation that accrues at the rate of k-u%.
- If we borrow one unit now, we have to compensate A by providing $e^{(k-u)T}$ units at date T.
- Equivalently, for every $e^{(k-u)T}$ units that we take today we owe 1 unit at date T.

- So we go long 1 forward contract, and we borrow $e^{(k-u)T}$ units, and sell them.
- At date T, we purchase 1 unit for the forward price, and return 1 unit to the borrower. The net profit is:
  \[
  \text{Profit} = e^{(k-u)T}S(0)e^r - FO(0) = S(0)e^{(r+u-k)T} - FO(0)
  \]
- Profit if $S(0)e^{(r+u-k)T} > FO(0)$
- Hence to avoid riskless arbitrage:
  \[
  FO(0) > S(0)e^{(r+u-k)T}
  \]
Commodity Forward Prices.

\[ S(0)e^{(r+u-k)T} < F(0) < S(0)e^{(r+u)T} \]

- The upper bound is easy to obtain. The lower bound is a problem, since the convenience yield is not known.

- Forward prices on commodities can have such wide upper and lower bounds, that little can be said about these prices from arbitrage arguments alone!
- We need a deeper model.

Bounds on Forward Prices
Interpreting the Convenience Yield

- Let $k$ be the convenience yield. You can view this like a dividend yield, but it is not observable!
- Then, we will write:
  $$F(0) = S(0)\exp[(r+u-k)T]$$
- When will $k$ be large?
- When will $k$ be small?
- Can $k$ depend on the time horizon? (ie. Different $k$ values for different futures contracts.)

The Implied Convenience Yield

- If the convenience yield was $k\%$, then
  $$F(0) = S(0)e^{(r+u-k)T}$$
  - In practice the forward prices are given. So we can imply out the average convenience yield over the period $[0,T]$. 
**Example**

On April 1\(^{st}\), \(S(0) = 1.96\), \(u = 1\%\), \(r=9\%\)

<table>
<thead>
<tr>
<th>Settle Date</th>
<th>May</th>
<th>July</th>
<th>Sept</th>
<th>Dec</th>
<th>March</th>
</tr>
</thead>
<tbody>
<tr>
<td>Futures Price</td>
<td>1.95</td>
<td>1.92</td>
<td>1.87</td>
<td>1.89</td>
<td>1.89</td>
</tr>
<tr>
<td>Upper Bound</td>
<td>1.976</td>
<td>2.001</td>
<td>2.043</td>
<td>2.095</td>
<td>2.148</td>
</tr>
<tr>
<td>Implied k</td>
<td>16.13%</td>
<td>18.24%</td>
<td>21.28%</td>
<td>15.46%</td>
<td>13.96%</td>
</tr>
</tbody>
</table>

**Implied Convenience Yields**

- Consider maturity \(s\) and \(t\) contracts with \(s<t\)
- \(F_{0_t}(0) = F_{0_s}(0)e^{(r+u-k)(t-s)}\)

- Why is this true?
- We can imply out convenience yields over specific periods.
Example (Continued)
On April 1st, \( S(0) = 1.96, u = 1\%, r=9\% \)

<table>
<thead>
<tr>
<th>Settle Date</th>
<th>May</th>
<th>July</th>
<th>Sept</th>
<th>Dec</th>
<th>March</th>
</tr>
</thead>
<tbody>
<tr>
<td>Futures Price</td>
<td>1.95</td>
<td>1.92</td>
<td>1.87</td>
<td>1.89</td>
<td>1.89</td>
</tr>
<tr>
<td>Upper Bound</td>
<td>1.976</td>
<td>2.001</td>
<td>2.043</td>
<td>2.095</td>
<td>2.148</td>
</tr>
<tr>
<td>Implied ( k )</td>
<td>16.13%</td>
<td>18.24%</td>
<td>21.28%</td>
<td>15.46%</td>
<td>13.96%</td>
</tr>
</tbody>
</table>

Forward Prices, Spot Prices, and Forecasts

- Expect future prices to be high:
  - Sellers will increase their inventories so as to capture the high prices.
  - Buyers will want to buy more today.
  - Buyers will demand more from sellers who prefer to sell less.
  - These conflicting positions will drive spot prices up
  - Price expectations are an important part of the formation of spot prices.
- With inventory, changes in future expectations have impact on current prices.
Price Variations and Storage Issues

- As storage costs increase, the connections between expected future prices and current prices diminish.
  - With high storage costs the alternative of holding inventories for future sale is less attractive than if storage costs are negligible.
  - As storage costs increase, expectations of higher future prices will have less impact on current spot prices.
  - As storage costs increase, the price variation over time in the forecasts increase.
  - As storage costs increase, actual price variations over time increase.

Example: Electricity

- Electricity is difficult to store. Expected price forecasts vary by the hour.
- Actual prices fluctuate widely.
- Cost of carry model and reverse cost of carry model do not hold.
What is the link between forward prices and expected spot prices for non-storables?

Suppose \( k \) is the expected return required by investors on an asset. \( (k = r + \text{risk premium}) \)

We could invest \( \text{Fexp}(-rT) \) today to get \( S(T) \) at date \( T \).

How would you do this?

Hence

\[ \text{Fexp}(-rT) = \mathbb{E}[S(T)]\exp[-kT] \]

No systematic risk?

Positive systematic risk?

Negative systematic risk?
Futures Price For Nonstorable Commodities

- E[S(T)] = 100
- F(0) = 100
- Perhaps a speculator would enter a forward contract at 90. Expect to make a $10 profit.
  - What if F(0) = 99
  - The speculator expects to make $1, but risk....
- Futures prices may not be unbiased estimators of future spot prices.

Normal Backwardation

- If all hedgers were SHORT, all speculators, LONG.
  - Hedgers will pay a premium to reduce risk
  - Speculators will require a premium to accept risk.
  - F(0) < E[S(T)]
- This market is called normal backwardation.
### A Contango Market

- If all hedgers were LONG, all speculators, SHORT
- Hedgers will pay a premium to reduce risk
  - Speculators will require a premium to accept risk.
  - $F(0) > E[S(T)]$
- This market is known as CONTANGO.

- If $F(0) = E[S(T)] = E[F(T)]$...
  - Futures prices are martingales

### Expected Spot Prices and Forward Prices

- The risk premium equals the difference in prices (expected spot less forward), as a percentage of the forward.

![Image of Spot Forecasts and Forward Curve with Maturity axis](image-url)
There is no theoretical reason for forward prices to act as an unbiased estimator of future expected prices.

The risk premium may be positive or negative!

The risk premium may start out big positive, say, and then decline to zero at the expiration date.

The risk premium exists due to risk aversion. Producers (sellers) are generally risk averse. They may have debt obligations associated with their capital stock. They need stable cash flows.

So, they may be willing to accept forward prices that are less than expected spot prices.

---

Buyers may also have a desire to avoid price risk.

They may prefer to pay a premium for price security. This implies that forwards might be higher than expected prices and risk premiums would be negative.

Relative desire for price security between these groups that will determine the sign of the risk premium.
Speculators

- Speculators are also present. They are concerned about forward prices relative to expected future prices.
- Does the strategy of buying forward and then selling at spot prices increase the total risks of a well diversified portfolio?
  - Yes—speculators need a positive risk premium. Forward will be below expected prices.

The Market Price of Risk

- The Risk Premium can be decomposed into two parts:
  - Volatility (or the amount of risk)
  - The price per unit of risk.
- The price per unit of risk is called the market price of risk.
- Estimating the market price of electricity risk is difficult.
Summary

- Forward prices and futures prices are close to each other.
- Futures give cash flows right away, forward do not. This has hedging implications.
- For securities, the cost of carry model and the reverse cost of carry model gives us forward prices.
- Market imperfections lead to bounds on prices.
- For commodities storage costs and convenience yields must be considered.
- The upper bound is easily computable, but the lower bound depends on the convenience yield.

Summary

- For commodity forwards and futures there is some flexibility as to where the forward prices can fall.
- That is, arbitrage alone, cannot identify the theoretical price, and we need richer models.
- When storage costs are high, forward prices and indeed spot prices and their forecasts can be more volatile. Cheap storage has the effects of smoothing prices over time.
- Implied convenience yields can be extracted from futures prices and can be informative.
Summary

- In general commodity forward curves will reflect seasonality since the convenience yields will fluctuate according to aggregate inventory levels.
- Electricity is not readily storable, and forward prices here represent a challenge.
- For non-storables, forward prices are linked to expected future spot prices. However, they are not unbiased estimators.

Summary

- For electricity prices, we may expect spikes and other “weird” phenomenon. For storables, as future expectations change, a ripple effect is felt through to current prices. Price changes should therefore be more smooth.
- If storage costs increase, then price volatility should increase.