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The Stochastic Process of Asset Prices

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Overview

- Objectives of the Study.
- Literature Review.
- The Basic Models.
- Data.
- Model Implementation.
- Experimental Design.
- Empirical Results for PCA Models.
- Empirical Results for 4-Parameter Models.
- Conclusions.

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Objective of the Study

The primary objectives are:

- To identify the key factors that effect stock price changes;
- To explain the geometric Wiener process for representing prices;
- To learn how to simulate stock prices from a geometric Wiener process; and
- To introduce the binomial lattice model as an approximation to a geometric Wiener process.

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Efficient Markets

- The keen competition among the participants ensures that new information regarding securities is rapidly absorbed and reflected in prices.
- If security prices do fully reflect all available information, the market is said to be *efficient*.
- If the market were not efficient, then some information might not be rapidly used in the setting of prices.
- Based on these patterns, trading strategies yielding abnormal rates of return could be derived.
- Unfortunately, as soon as these strategies became apparent to investors, the information would soon be more efficiently used and abnormal profits would soon be eliminated.

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- Consider the sequence prices, $S_0, S_1, S_2....$ observed with gaps Δt. (Eg $\Delta t = 1/52$ year) or $\Delta t = 1/365$
- Let $R_t = S_t/S_{t-1}$ be the price relative over the tth time increment, which is of width Δt.
- Then, given the original stock price is S_0, future stock prices can be represented as follows:

$$\begin{aligned} S_1 &= S_0 R_1 \\ S_2 &= S_1 R_2 = S_0 R_1 R_2 \\ S_n &= S_0 R_1 R_2 \ldots R_n. \end{aligned}$$

- Let $r_t = ln(R_t)$ be the logarithmic return over the tth time increment.
- We shall assume that the distribution of logarithmic returns are independent of the level of the stock price.
- This implies, for example, that a 10% or greater logarithmic return over a time increment is as likely for a stock priced at \$10, as it is for a stock priced at \$100.

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- In addition, we shall assume that the sequence of logarithmic returns are independent of each other.
- If the sequence of returns were not independent, then the history of the returns would provide useful information about the future logarithmic returns, and this may be inconsistent with a weak form efficient market.

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- Let α represent the expected value of the logarithmic returns over one year.
- For example, if $\alpha = 20\%$, then we would expect the logarithmic return over one year to be 20%.
- The expected return over a 6 month period would be 10%, and over a period of length Δt years would be $20\Delta t\%$
- Like the expected reward, the uncertainty in the logarithmic return should increase as the holding period increases. For example, the logarithmic return over one day is likely to be much less variable than the return over one year.
- The magnitude of this discrepency is captured by the standard deviation per unit of time.
- An annual standard deviation of 30% per year means that while $\alpha = 20\%$ is expected, fairly large deviations from this value are possible. Indeed, for the normal distribution, about 2 standard deviations around the mean capture over 90% of the possible outcomes. Hence, in one year, we should be confident that the logarithmic return is $20\% \pm 60\%$. A return outside this interval would be considered rare.
- In finance, the standard deviation is referred to as the volatility.

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- The volatility of return expands over time. As an example, the volatility over a time period of two years will be much larger than the volatility over one year.
- The most common assumption of models of stock price behavior assumes the volatility expands with the square root of the time period. Hence, if the volatility of returns is 30% per year, the volatility over a period of Δt years is $30\% \times \sqrt{\Delta t}$.
- Assume that the distribution for the logarithmic return over any period is normal.

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- In summary, then, our model of stock price behavior assumes the logarithmic returns in successive periods of width Δt years are independent normal random variable with mean $\alpha \Delta t$, linear in the time increment, and standard deviation, $\sigma\sqrt{\Delta t}$, which grows with the square root of the time increment.

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- Let $\{Z_t \mid t = 1, 2..\}$ represent a sequence of independent standard normal random variables, with mean zero and standard deviation, 1. To standardize any normal random variable, we subtract the mean, and divide by the standard deviation. Hence, we obtain:

$$Z_t = \frac{r_t - \alpha\Delta t}{\sigma\sqrt{\Delta t}}.$$

The logarithmic return can then be expressed as:

$$r_t = \alpha\Delta t + \sigma\sqrt{\Delta t}Z_t. \tag{1}$$

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- Since the logarithmic return is a normal random variable it could be any number, positive or negative.
- Now consider the price relative, $R_t = e^{r_t}$. Exponentiating any number, positive or negative, leads to a positive value. Hence, while the distribution of the logarithmic return is normal, the distribution of the price relative is not.
- The statistical distribution is called the *lognormal distribution*.

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- The expected value of the price relative over any time increment Δt can be shown to be

$$E(R_t) = e^{\mu \Delta t}$$

where $\mu = \alpha + \sigma^2/2$.

- To avoid confusion over terminology regarding means, whenever we refer to α, we shall emphasize that it is the mean of the *logarithmic* or *continuously* compounded returns, while μ will be referred to as the *expected growth rate* or just the *expected return*.

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Example

- A non-dividend paying stock, priced at \$100 has logarithmic returns that are normally distributed. The expected logarithmic return is 20% per year,($\alpha = 0.20$) and the volatility is 30% per year ($\sigma = 0.30$).
- Over a six month period the expected logarithmic return is 10%, and the volatility is $0.30\sqrt{0.5} = 0.212$.
- The probability of the logarithmic return exceeding 18% over six months ($\Delta t = 0.5$) can be computed by standardizing the random variable as follows.
  $P(r_1 > 0.18) = P(Z_1 > \frac{0.18-0.10}{0.212}) = P(Z > 0.377)$.
  Using standard normal tables, this probability is 0.35.

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- The expected return of the stock is $\mu = \alpha + \sigma^2/2 = 0.245$, or 24.5% per year.
- The distribution of the price relative over any six month period is lognormal with expected value $E(R_t) = e^{0.245 \times 0.5} = 1.1303$.
- The expected stock price after one year is $S_0 e^{\mu} = \$127.76$ and after two years is $S_0 e^{2\mu} = \$163.23$. Hence the expected stock price change in the first year is \$27.76 while the expected stock price change over the second year is $163.23 - 127.76 = \$35.47$.
- While logarithmic returns in successive periods are independent identically distributed normal random variables, actual price changes are not. This property seems reasonable. For example, a price change of \$10, is more likely for a stock priced at \$100, then for a stock priced at \$10.

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The Geometric Wiener Process

- A stock follows a Geometric Wiener process if:

  (i) Over any time increment, Δt, the distribution of logarithmic returns is normally distributed with mean $\alpha \Delta t$, proportional to the time increment , and volatility, $\sigma\sqrt{\Delta t}$, proportional to the square root of the time increment.

  (ii) The logarithmic returns over non overlapping time increments are independent.
- Our representation of the logarithmic returns for a stock over any arbitrary period of time, Δt, is given by

$$r_t = \alpha \Delta t + \sigma\sqrt{\Delta t} Z_t$$

Over the tth time increment the logarithmic return is $r_t = ln(S_t/S_{t-1})$. Substituting this expression into the above

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equation we obtain

$$ln(S_t/S_{t-1}) = \alpha\Delta t + \sigma\sqrt{\Delta t}Z_t$$

Hence, the stock price at period t is completely determined by the stock price at period t and the standard normal variable, Z_t. In particular,

$$S_t = S_{t-1}e^{\alpha\Delta t+\sigma\sqrt{\Delta t}Z_t} \tag{2}$$

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Simulating Stock Price Paths

- Assume α and σ are given, as well as Δt. Then, given the initial stock price, S_0, we use equation (2) to generate S_1 which corresponds to the price at date Δt.
- This of course requires us to use the first random number. We then use this new price, and the second random number to generate S_2 at date $2\Delta t$.
- Repeating this procedure over n time increments, leads to a path of prices up to date $n\Delta t$.

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- In Excel, under "Tools", "Data Analysis", "Random Number Generator", you will find the normal random number generator.
- It will ask you for
  1. the mean and standard deviation, which is 0 and 1 in our case;
  2. the number of different random variables to generate, which in our case is 1;
  3. the length of the sequence, say 12, if you want 12 numbers;
  4. and the seed.

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Parameters	
S(0)	100
μ	0.10
σ	0.3
Δt	1/52
$\alpha = \mu - \sigma^2/2)$	0.055

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Time	Zvalue	StockPrice
0	-	100
1	-2.939	80.08
2	1.154	87.50
3	-0.321	85.49
4	0.662	89.99
5	1.012	97.28
6	1.741	111.16
7	0.613	116.58
8	0.097	117.57
9	0.988	126.86
10	2.663	155.47
11	-1.011	144.13
12	1.100	156.85

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- If $\sigma = 0$, then there would be no uncertainty and this expected price path would be followed. As σ expands, uncertainty increases and it becomes more likely that price paths deviate significantly from the expected trajectory.
- Simulation is an important technique for option pricing that will be used in future chapters.

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Estimation of the Parameters of a Geometric Wiener Process

- daily closing prices of a nondividend-paying stock over a period of 12 consecutive weeks. The $n = 12$ logarithmic returns are computed by $r_t = ln(S_t/S_{t-1})$.

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Week	Stock Price	Price Relative	Log Return
0	100.00		
1	80.08	0.801	-0.222
2	87.50	1.093	0.089
3	85.49	0.977	-0.023
4	89.99	1.053	0.051
5	97.28	1.081	0.078
6	111.16	1.143	0.133
7	116.58	1.049	0.048
8	117.57	1.008	0.008
9	126.86	1.079	0.076
10	155.47	1.226	0.203
11	144.13	0.927	-0.076
12	156.85	1.088	0.085
		Weekly Average	0.0375
		Weekly Std. Dev.	0.109
		Annual Average	1.951
		Annual Std. Dev.	0.783

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- Assuming the weekly logarithmic returns are independent and come from the same statistical distribution, then the weekly logarithmic mean and variance can be estimated by the following:

$$\begin{aligned} \hat{\alpha}\Delta t &= \frac{1}{n}\sum_{i=1}^{n} r_i \\ \hat{\sigma}^2\Delta t &= \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{\alpha})^2 \end{aligned}$$

  where $\Delta t = 1/52$ and $\hat{\alpha}$ and $\hat{\sigma}$ are the annualized mean and volatility estimators.

- In the above example, we annualized the variance by multiplying the weekly variance by 52. Our final volatility is given by $\hat{\sigma} = 0.783$ or 78.3%.

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- If daily data is used, then the variance is annualized by multiplying by the number of days (365) in the year.
- Actually, it may be better to annualize the variance by multiplying by the number of business days (280) in the year. The reason for this is due to the lack of price volatility over the weekends.

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- Empirical Evidence of Stock Return Behavior
- Binomial Approximation of a Binomial Process
- Other Comments