Creating an Abstract Concept of Addition

Having used abstraction to create a set V of objects that includes n-vectors and matrices as special cases, you now need a method for performing operations on those objects. For example, as with n-vectors and matrices, you would like to be able to "add" elements u and v of V. To do so, consider using a general binary operation on V, denoted by $\oplus$, that combines two elements u and v of V to create a new element of V, namely,

$$u \oplus v \tag{4.1}$$

The binary operation in (4.1) is called a **closed** binary operation on the set V, meaning that for any two elements u and v in V, $u \oplus v \in V$.

Observe that the details of how $\oplus$ in (4.1) is used to combine u and v are not specified. However, when working with a specific set V of objects, such as n-vectors, you need to specify precisely how the operation $\oplus$ is used to combine two elements.

Creating an Abstract and an Axiomatic System

Abstraction has led to a unification of addition of n-vectors and matrices consisting of a set V together with a closed binary operation, $\oplus$, on V.

> **Mathematical Thinking Process**
>
> The pair $(V, \oplus)$ is an example of an **abstract system**, meaning a set together with one or more ways to perform operations on the elements of the set.

The advantage of abstraction is that you can unify different mathematical items together in the framework of a single set. As you have just seen, a disadvantage of abstraction is that you lose properties of the specific items that give rise to the abstraction. As another example of this disdvantage, recall that 0 is an n-vector that is special with regard to adding n-vectors because,

$$\text{for all } n\text{-vectors } u, u + 0 = 0 + u = u \tag{4.2}$$

Likewise, the zero matrix, 0, is special with regard to adding matrices because,

$$\text{for all matrices } A, A + 0 = 0 + A = A \tag{4.3}$$

When you use abstraction to unify n-vectors and matrices in the set V of objects, you lose the existence of this special "zero" item. You can overcome this deficiency by *hypothesizing* the existence of a special element in V that has the same desirable properties as the corresponding element in the special cases. For example, you can hypothesize the existence of a special element $0 \in V$. For this element to have the same desirable properties as the zero vector in (4.2) and as the zero matrix in (4.3), 0 should satisfy the following property with respect to the operation $\oplus$ on V:

For all elements $v \in V; v \oplus 0 = 0 \oplus v = v$

In summary, although abstraction results in losing properties that apply to the special cases, one way to overcome this deficiency is by creating **axioms**, which are properties that are assumed to hold in the corresponding abstract system. For instance, in this example, you create the following axiom for the abstract system $(V; \oplus)$ to ensure the existence of an element of V that has the same properties as the zero vector and the zero matrix:

$$\text{There is a } 0 \in V \text{ such that for all } v \in V; v \oplus 0 = 0 \oplus v = v \tag{4.4}$$

You are free to choose the specific axioms to include with the abstract system. Which ones you choose depend on what properties of the special cases you want to study, the kind of results you eventually want to obtain about the abstract system, and more. In any event, when finished, the result is the following system.

Mathematical Thinking Process

An **axiomatic system** is an abstract system together with a list of axioms that are assumed to hold true.

The remainder of this section is devoted to the development of additional operations and axioms to create an axiomatic system that has the same operations and properties as the special cases of n-vectors and matrices.