Figure 4.5: Examples of Vectors that Open Up Properly

4.3.3 Linearly Dependent and Independent Vectors

For n-vectors, the property needed to ensure a unique solution to (4.20) is that of vectors \opening up properly." For example, the vectors u and v in Figure 4.5(a) and in Figure 4.5(b) open up properly. In contrast, the vectors u and v in Figure 4.6(a) and in Figure 4.6(b) do not open up properly. This is because the vectors in Figure 4.6(a) lie on top of each other and the vectors in Figure 4.6(b) point in exactly opposite directions. The objective now is to create a mathematical de¯nition of what it means for vectors to \open up properly" by translating the associated visual image to symbolic form.

One approach is to develop a de¯nition for the property satis¯ed by the vectors in Figure 4.5. Alternatively, you can ¯rst develop a de¯nition for the property of \not opening up properly," satis¯ed by the vectors in Figure 4.6, and then write the negation of that de¯nition to capture the property of the vectors in Figure 4.5.

Linearly Dependent Vectors

Following the latter approach, try to identify similarities between the pair of vectors in Figure 4.6(a) and the pair in Figure 4.6(b). For example, in Figure 4.6(a), u points in the same direction as v but has a di®erent length. Recall from Section 1.2.2 that you can change the length of v (without a®ecting the direction) by multiplying v by a nonnegative real

Figure 4.6: Examples of Vectors that Do Not Open Up Properly

number, say, a ¸ 0. Thus, by an appropriate choice of the real number a ¸ 0, you can make the vectors u and av equal. You can translate this observation to the following symbolic form using the quanti¯er there is:

There is a real number a ¸ 0 such that u = av: (4.21)

Looking now at Figure 4.6(b), you will notice that u points in the opposite direction to v. In this case, by multiplying v by an appropriate choice of the nonpositive real number a · 0, you can again make u and av equal. This observation translates to the following symbolic form:

There is a real number a · 0 such that u = av: (4.22)

Can you create a single statement that includes both (4.21) and (4.22) as special cases? One such approach is to allow a to be any real number | positive, negative, or 0. Thus, a uni¯cation of (4.21) and (4.22) is the following:

There is a real number a such that u = av: (4.23)

Generalizing the Definition to 3-Space. The next step is to generalize (4.23) when the vectors lie in 3-space. Indeed, (4.23) is still valid when u and v are vectors in 3-space that lie on top of, or opposite to, each other. However, what if you are working with a third vector, w, in 3-space, such

Figure 4.7: Three Vectors in R^3 that Do Not Open Up Properly

as the one in Figure 4.7? You might write the following statement for this property:

$$\text{There are real numbers } a \text{ and } b \text{ such that } u = av + bw \tag{4.24}$$

Is the property in (4.24) correct for all groups of three vectors in 3-space that do not open up properly? Only by extensive trials with other special cases of u, v, and w will you discover that (4.24) is not necessarily correct in all cases. For example, the vectors in Figure 4.8 do not satisfy (4.24) because $av + bw$ always points in the same direction as v and w. Rather, the vectors in Figure 4.8 satisfy the following property:

$$\text{There are real numbers } a \text{ and } b \text{ such that } v = au + bw \tag{4.25}$$

The vectors in Figure 4.8 also satisfy the property that

$$\text{there are real numbers } a \text{ and } b \text{ such that } w = au + bv \tag{4.26}$$

In fact, at least one of (4.24), (4.25), or (4.26) is always true for vectors u, v, and w in 3-space that do not open up properly. The challenge is to write a single statement that covers all three of the special cases in (4.24), (4.25), and (4.26). Such a uni¯cation requires cleverness by moving all vectors to the same side of the equality sign and then realizing that there are real numbers associated with each vector, that is,

$$\text{There are real numbers } a; b, \text{ and } c \text{ such that } au + bv + cw = 0 \tag{4.27}$$

Figure 4.8: Three Other Vectors in R^3 that Do Not Open Up Properly

However, the real numbers in the special cases are not just any real numbers — they have some special properties: in (4.24), $a = 1$; in (4.25), $b = 1$; and in (4.26), $c = 1$. One way to capture this fact is to require in (4.27) that at least one of the real numbers be 1. An alternative, but equivalent, way to say this (as you are asked to verify in Exercise 25) is the following:

$$\text{There are real numbers } a, b, c, \text{ not all 0, with } au + bv + cw = 0 \quad (4.28)$$

If $a \neq 0$, for example, then statement (4.28) is equivalent to saying that u is a linear combination of v and w.

Generalizing the Definition to n-Space. The next step is to generalize (4.28) to the case where the vectors lie in n-space. So suppose you have a group of k vectors in n-space that do not open up properly. Once again, subscript and superscript notation is helpful. Let each of $v_1, \ldots, v_k$ be an n-vector. By introducing the term "linearly dependent" for "not opening up properly" and by generalizing (4.28) in the natural way, you obtain the following definition.

Definition 4.6 The n-vectors $v_1, \ldots, v_k$ are *linearly dependent* if and only if there are real numbers $t_1, \ldots, t_k$, not all zero, such that

$$t_1 v_1 + \cdots + t_k v_k = 0$$