Theorem 1.1 If u, v, and w are n-vectors and s and t are real numbers, then the following properties hold:

(a) $u + v = v + u$.

(b) $(u + v) + w = u + (v + w)$.

(c) $u + 0 = 0 + u = u$.

(d) $u + (-u) = 0$.

(e) $s(tu) = (st)u$.

(f) $s(u + v) = (su) + (sv)$.

(g) $(s + t)u = (su) + (tu)$.

(h) $1u = u$.

(i) $0u = 0$ (the real number 0 times the vector u is the vector 0).

Proof. A proof is an argument designed to convince someone that the following type of mathematical statement, called an *implication*, is true:

if p is true, then q is true

or, more simply,

if p, then q

where p is a statement referred to as the *hypothesis* and q is a statement called the *conclusion*. For example, in Theorem 1.1(a), you have the following implication with hypothesis p and conclusion q:

if $\underbrace{u \text{ and } v \text{ are } n\text{-vectors}}_{p}$, then $\underbrace{u + v = v + u}_{q}$

A more detailed discussion of proofs is given in Appendix A, which you should read as needed and when indicated in this book.

Mathematical Thinking Process

There are various proof techniques you can use to establish that the implication "if p, then q" is true. With one of them, referred to here as the forward-backward method, you can assume that p is true. Your objective is to use this assumption to show that q is true (see Appendix A.1 and Appendix A.2). An effective approach to doing so is to work backward from the conclusion by asking yourself the key question: "How can I show that q is true?" After asking the key question, you must provide an answer. One common way to do so is to use a definition.

To illustrate working backward with Theorem 1.1(a), in which the conclusion is $u + v = v + u$, a valid key question is "How can I show that two n-vectors (namely, $u + v$ and $v + u$) are equal?" This same question is appropriate for the conclusion in each part of Theorem 1.1.

One answer to the foregoing key question is to use Definition 1.2 to show that each component of one n-vector is equal to the corresponding component of the second n-vector. You will now see how this is done. You will also find it helpful to use visual images of vectors and their operations in two dimensions to motivate the following proofs.

(a) To see that the two vectors $u + v$ and $v + u$ are equal, use Definition 1.2 to show that each component of $u + v$ equals the corresponding component of $v + u$, that is, you must show that

$$\text{for each } i = 1, \ldots, n, (u + v)_i = (v + u)_i$$

To this end, you have that

$$\begin{aligned} u + v &= (u_1 + v_1, \ldots, u_n + v_n) && \text{(definition of } u + v\text{)} \\ &= (v_1 + u_1, \ldots, v_n + u_n) && \text{(commutative property of addition of real numbers)} \\ &= v + u && \text{(definition of } v + u\text{)} \end{aligned}$$

(b-d) The proofs of these properties are left to Exercise 23.

(e) Again, use Definition 1.2 to show that the vector $s(tu)$ is equal to the vector $(st)u$. This is accomplished with the following steps, which are justified by the definition of multiplying a vector by a real number:

$$\begin{aligned} s(tu) &= s(tu_1, \ldots, tu_n) \\ &= (s(tu_1), \ldots, s(tu_n)) \\ &= ((st)u_1, \ldots, (st)u_n) \\ &= (st)(u_1, \ldots, u_n) \\ &= (st)u \end{aligned}$$

(f-i) The proofs of these properties are left to Exercise 23.

This completes the proof.

QED