1.3.3 Unifying the Equations of a Plane

In Section 1.3.2, you were introduced to the following two ways to represent a plane in n-space:

$$z = m_1 x_1 + \cdots + m_{n-1} x_{n-1} + b \tag{1.32}$$

$$a_1(x_1 - x_1^0) + \cdots + a_n(x_n - x_n^0) = 0 \tag{1.33}$$

It would be convenient if there were one way to represent the equation of a plane that includes both (1.32) and (1.33) as special cases.

> Mathematical Thinking Process
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> The process of creating this new, encompassing representation is an example of another mathematical technique called unification. This technique involves combining two or more concepts (problems, theories, and so on) into a single framework from which you can study each of the special cases.

To illustrate, unification is used now to combine the special cases in (1.32) and (1.33) into a single equation for representing a plane in n-space.

Identifying Similarities and Differences

The first step in unification is to identify similarities and differences between the special cases in an attempt to find the commonalities. For instance, (1.32) and (1.33) exhibit the following similarities and differences.

Similarities in (1.32) and (1.33)

1. Both equations involve n variables: z and $x_1, \ldots, x_{n-1}$, in (1.32) and $x_1, \ldots, x_n$, in (1.33).
2. Each of the variables in both equations is multiplied by a known constant. For example, in (1.32), x_1 is multiplied by m_1 and in (1.33), x_1 is multiplied by a_1.
3. Both equations involve other known constants: b, in (1.32) and $a_1 x_1^0, \ldots, a_n x_n^0$, in (1.33).

Differences in (1.32) and (1.33)

1. The left and right sides of the two equations have different forms. For example, the right side of (1.32) contains variables together with the constant b. The right side of (1.33) contains only the constant 0.
2. The specific constants that multiply the variables are different in the two equations.

3. One of the variable names in (1.32) differs from that in (1.33). Specifically, (1.32) contains the variable z and (1.33) contains x_n.

The objective now is to find ways to eliminate the differences so that you can unify (1.32) and (1.33) into a single equation.

Eliminating the Differences

You can eliminate the first difference by rewriting (1.32) and (1.33) so that all terms involving variables are on the left side of the equality signs and all terms involving constants are on the right side. For instance, moving all terms involving variables in (1.32) to the left side results in

$$-m_1x_1 - \cdots - m_{n-1}x_{n-1} + z = b \tag{1.34}$$

Likewise, removing the parentheses in (1.33) and moving all constant terms to the right side results in

$$a_1x_1 + \cdots + a_nx_n = a_1x_1' + \cdots + a_nx_n' \tag{1.35}$$

Now, in both (1.34) and (1.35), all variables are on the left side of the equality sign and only known constants are on the right side.

The second difference identified above is overcome by introducing new notation, as done, for example, in the following unification of (1.34) and (1.35):

$$c_1y_1 + \cdots + c_ny_n = d \quad \text{(unification)} \tag{1.36}$$

In the unification in (1.36), the new symbols $y_1, \ldots, y_n$ are used to represent the n variables from the special cases in (1.34) and (1.35). The symbols c_1 through c_n in the unification represent known constants that multiply the corresponding variables in the special cases. The symbol d in the unification represents a known value on the right side of the equalities in the special cases.

As you have seen, the use of appropriate notation is important. To avoid confusion, all symbols used in the unified equation (1.36) are different from those in the special cases. Often, however, mathematicians use overlapping notation, that is, the same symbol is used more than once, with a different meaning in each case. For example, you can write (1.36) equally well as

$$a_1x_1 + \cdots + a_nx_n = b \tag{1.37}$$

In this case, the symbols $a_1, \ldots, a_n$ in (1.37) overlap with those same symbols in (1.33) and the symbol b in (1.37) overlaps with that same symbol in (1.32). Also, the symbols $x_1, \ldots, x_{n-1}$ in (1.37) overlap with those same symbols in (1.32) and (1.33). When this overlapping notation arises, be sure to keep the meaning of the symbols straight in your mind.